THE SIZE OF THE UNIVERSE

ATTEMPTS AT A DETERMINATION OF THE CURVATURE RADIUS OF SPACETIME

BY

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PREFACE

The principal object of this book is the determination of the invariant curvature radius of spacetime, and thus also of space considered as its normal section. In an introductory part the fundamental concepts associated with amorphous and, later on, metricized spacetime are explained and the elements of the tensor calculus are recalled. It is hoped that this will enable even those readers who are not skilled in Tensors and Relativity theory to follow freely the deductions. In the next place the two solutions of Einstein's amplified (cosmological) field-equations, one due to Einstein himself and the other due to de Sitter, are discussed. These are commonly referred to as the cylindrical and the pseudo-spherical or isotropic spacetime. Ample reasons are given for rejecting the cosmology based upon the former and for concentrating one's attention upon that based on the latter solution. Accordingly, after a digression about Dr. Hubble's investigation of extra-galactic nebulae in connexion with the cylindrical world, the remainder of the book is exclusively devoted to isotropic spacetime. Its main properties are treated with considerable detail. More especially, the Doppler-effect formula, for a light-source and an observer in inertial motion, is derived from the line-element of this spacetime. Unlike de Sitter's red-shift due to distance (a second-order effect), the influence of

distance upon the radial, spectroscopic velocity of a star is shown to be a first-order effect, leading to an observable correlation between radial velocity, regardless of its sign, and distance. This formula holds for any, generally, oblique motion of the star. A statistical method of application of the formula is developed, by which the two unknown star-constants can, under certain assumptions, be eliminated, leading to a formula for the curvature radius in terms of distances and radial velocities of two groups of stars. This is applied to eighteen globular clusters and the Magellanic Clouds, and the preliminary value of the curvature radius, of the order of 7.10^{12} astronomical units, thus obtained is discussed in some detail. Some general implications of a finite curvature radius are then developed and illustrated on numerical examples based on the value just mentioned. The most remarkable among these implications is the existence of a certain critical distance or 'critical radius' associated with every mass or system of masses, say galaxy of stars. It is proportional to the cube-root of the mass and to the power $\frac{2}{3}$ of the curvature radius of spacetime. At this 'critical' distance the gravitation is just balanced by a quasi-centrifugal action. The latter can be represented as if due to a spin about the mass-centre, the period of this spin, a universal constant called the 'cosmic day', being equal to twice the total length of a straight line divided by the light velocity. In the case of a

spherical galaxy of stars the critical radius of its total mass is the upper limit to its semi-diameter: when this exceeds the critical radius, the galaxy ceases to be permanent, its members deserting it on quasi-hyperbolic orbits. Equivalently the criterion of permanency is reducible to a 'critical density' of mass distribution (discrete or continuous), which is a universal constant. The criterion applied to our own galaxy, the Milky Way, shows that it is much too inflated to be permanent.

The main text is followed by Miscellaneous Notes. In five of these some special points are treated. In the three remaining ones, Note 3, Note 4, and Note 8, the chief object of the book, viz. the determination of the curvature radius, is again taken up. The stars here consulted are much nearer than the clusters considered in the text. At the same time they are far more numerous and offer also the additional advantage that, besides the radial velocities, their proper motions or transversal velocities are known. Owing to this circumstance one of the two star-constants (the perihelion distance) can be eliminated without any arbitrary assumption. This, together with the greater number of data, gives to the results here obtained a much higher degree of reliability. Moreover, the three batches of stars, 24 Cepheids, 35 stars of the O type, and 459 mixed stars, treated in these Notes independently of each other, lead materially to the same value of the radius, viz. 3 to

4.10^{11} astronomical units. For the present at least, the weighted mean of these three determinations can be accepted as the actual value of the curvature radius. The value derived from the clusters and the Clouds is some eighteen times greater, but, for reasons explained in the book, it is certainly less reliable. Moreover, the distances of the clusters and the Clouds are now being thoroughly revised by Prof. Shapley, and one can expect with confidence that, when they are corrected, these very distant celestial objects will materially corroborate the last-quoted result.

In spite of the many deficiencies of this monograph it is hoped that it will succeed in stimulating the reader's interest in a fundamental cosmological problem.

I gladly take the opportunity of expressing my gratitude to the officials and staff of the Clarendon Press for the care they have bestowed on this book.

L. S.

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CONTENTS

PART I

INTRODUCTORY

THE title of this book is a somewhat popularized, non-technical equivalent of 'the *Curvature Radius* of the Universe', the latter denomination being again a compromise abbreviation for the four-dimensional manifold known to modern physicists, astronomers, and mathematicians as *Spacetime* or 'world'.

The best approach to the subject to be discussed in this volume is to explain the meaning and the essential features of the concept of spacetime* itself. This, then, will be our first task.

Let P be a point (a 'place') in three-dimensional space. Imagine an event, say an explosion, a flash of light, happening at the point P at a given instant of time. *The event thus localized in space*, say by three coordinates

$$x_1,\ x_2,\ x_3$$

(any three mutually independent numerical data, relative to some conventional framework of reference), *and in time*, by the single number

$$t = x_4$$

(time, that is, reckoned from an arbitrarily chosen initial instant), is called a *worldpoint*. The world or spacetime is the class or the manifold, of which the worldpoints are the elements.

* Usually written 'space-time'.

We have, in passing, illustrated the worldpoint by a lightflash. A more useful illustration of a worldpoint is *the presence of a particle* (of matter) at a given place ($x_1 \, x_2 \, x_3$) at a given instant of time (x_4). Let us imagine a particle in continuous motion. This will give us a continuous* succession of worldpoints, i.e. of tetrads of numerical values of $x_1 \, x_2 \, x_3 \, x_4$. This continuous, one-dimensional sequence of tetrads $x_1 \, x_2 \, x_3 \, x_4$ or of events is called a *worldline*—the particle's own worldline, that is—it being assumed, of course, that it is possible to follow this speck of matter through its history. In the case of a body of finite dimensions we may think of a three-dimensional manifold of its constituent 'particles'. In the time-history of the body each of these points will describe a worldline, and the ensemble of these lines will form what is called a *worldtube*, a tube, that is, with a *sense* to it, viz. stretching, from past to future, which may be marked by an arrow. The shape of the (varying) cross-sections of such a four-dimensional tube will represent the complete history of the body under consideration.

To begin with, we may think of spacetime as a *non-metrical*, amorphous† manifold.

Later on metrics will be impressed upon it, the choice of the so-called fundamental or metrical *tensor*‡ being

* Or at least believed to be continuous, or better, assumed, for mathematical convenience, to be so.

† A name used by H. Weyl.

‡ This concept will be explained presently.

guided by considerations of a physical nature, viz. the peculiarities of the motion of free particles and of the propagation of light in 'empty' space.

To have a short and convenient symbol, let this non-metrical but, as we assume, continuous manifold be denoted by S_4, the suffix reminding us of the number of its dimensions (S_4 = four-space).

It goes without saying that to an (amorphous) S_4 thus conceived the very idea of 'distance' of points (worldpoints, events) as well as that of 'angle', say between two worldlines, or of 'volume' of a piece of a worldtube, is entirely foreign.

It is a remarkable fact that in spite of this a good number of interesting, relevant, properties of the manifold S_4, *the amorphous spacetime*, can be derived.

To familiarize ourselves with these properties it will be well to recapitulate here briefly the fundamentals of the *Tensor Calculus*, with which my readers are supposed to be acquainted.*

Let us imagine that we pass from the coordinates $x_1\, x_2\, x_3\, x_4$ (any coordinate system) to a system of new coordinates $x'_1\, x'_2\, x'_3\, x'_4$, some functions of $x_1\, x_2\, x_3\, x_4$, continuous with (for the present) their first deriva-

* Readers who have had no opportunity of acquiring the knowledge of this comparatively *modern* (1900) and powerful branch of mathematics may be referred to the English version of Prof. Levi-Civita's excellent book, *Lezioni sul calcolo differenziale assoluto*. A more condensed but sufficiently illuminating presentation is given in my article on *Tensor Analysis* in the 14th ed. of *Ency. Britannica*, 1929.

tives. Then the differentials dx_ι belonging to a pair of worldpoints, P (x) and Q $(x+dx)$, are, as is well known, transformed into

$$dx'_1 = \frac{\partial x'_1}{\partial x_1}dx_1 + \ldots + \frac{\partial x'_1}{\partial x_4}dx_4,$$

and similarly for dx'_2, dx'_3, and dx'_4. With the now generally adopted convention that each term in which an index occurs *twice* is to be summed over all the four values (1, 2, 3, 4) of the index, this set of elementary but very important transformation formulae can be written briefly

$$dx'_\iota = \frac{\partial x'_\iota}{\partial x_\kappa}dx_\kappa, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\iota = 1, 2, 3, 4$ (while the right-hand member has to be summed over $\kappa = 1, 2, 3, 4$). Vice versa, assuming that the Jacobian

$$J = \left|\frac{\partial x'_\iota}{\partial x_\kappa}\right| \equiv \begin{vmatrix} \frac{\partial x'_1}{\partial x_1} & & \\ & \cdot \; \cdot & \\ & & \frac{\partial x'_4}{\partial x_4} \end{vmatrix}$$

differs from 0 and ∞, we have

$$dx_\iota = \frac{\partial x_\iota}{\partial x'_a}dx'_a, \quad . \quad . \quad . \quad . \quad . \quad (1')$$

these four equations being the solution of (1) with respect to the dx_ι.

Now, every tetrad of magnitudes A^ι (where ι is just an index, not 'power') which—in general—may be functions of all four coordinates, and which

are transformed according to the same rule as dx_ι, i.e. into

$$(A^\iota)' = \frac{\partial x'_\iota}{\partial x_\kappa} A^\kappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is called a *contravariant vector* or *tensor* of *rank one.*

On the other hand, a tetrad of magnitudes B_ι which are transformed like the derivators $\partial/\partial x_\iota$, i.e. by the rule

$$B'_\iota = \frac{\partial x_\kappa}{\partial x'_\iota} B_\kappa, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is called a *covariant* vector, *lower* indices being used for such vectors and, more generally, tensors of any rank.

By an obvious extension of this concept, an array of $4^2 = 16$ magnitudes $C_{\iota\kappa}$ transformed into

$$C'_{\iota\kappa} = \frac{\partial x_\alpha}{\partial x'_\iota} \frac{\partial x_\beta}{\partial x'_\kappa} C_{\alpha\beta}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

will be a covariant, and one obeying the transformation rule

$$D'^{\iota\kappa} = \frac{\partial x'_\iota}{\partial x_\alpha} \frac{\partial x'_\kappa}{\partial x_\beta} D^{\alpha\beta}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

a contravariant tensor of *rank two.* If, in particular, $C_{\iota\kappa} = C_{\kappa\iota}$, the tensor is called *symmetrical*, and if $C_{\iota\kappa} = -C_{\kappa\iota}$ (so that $C_{\iota\iota} = 0$), we speak of an *antisymmetrical* or *skew* tensor, or 'six-vector' (since it has only six non-vanishing, mutually independent components). Similarly for the contravariant second-rank tensor. It is important to notice that symmetry, or skewness, is an invariant property; that is to say,

if $C_{\iota\kappa} = C_{\kappa\iota}$, then also, in any new system of coordinates, $C'_{\iota\kappa} = C'_{\kappa\iota}$. In much the same way tensors, covariant or contravariant, of the third and higher rank can be defined.

A *mixed* tensor of, say, rank two is characterized by the transformation rule

$$A'^{\kappa}_{\iota} = \frac{\partial x'_{\kappa}}{\partial x_{\beta}} \frac{\partial x_{\alpha}}{\partial x'_{\iota}} A^{\beta}_{\alpha}. \quad . \quad . \quad . \quad . \quad (6)$$

Similarly, one of the third rank, say, with e.g. one contravariant and two covariant indices, obeys the transformation rule

$$A'^{\lambda}_{\iota\kappa} = \frac{\partial x_{\alpha}}{\partial x'_{\iota}} \frac{\partial x_{\beta}}{\partial x'_{\kappa}} \frac{\partial x'_{\lambda}}{\partial x_{\gamma}} A^{\gamma}_{\alpha\beta}, \quad . \quad . \quad . \quad (7)$$

and so on.* Needless to say, a third-rank tensor consists, in general, of $4^3 = 64$ components.

The reader will see for himself that the sum of two tensors of same rank and kind, as e.g. $A_{\kappa} + B_{\kappa}$, or $C_{\iota\kappa} + D_{\iota\kappa}$, is again a tensor of the same rank and kind; while complexes of heterogeneous materials (misalliances, so to speak), as e.g. $A_{\kappa} + B^{\kappa}$ or $A_{\kappa} + C^{\iota\kappa}$, have no tensorial character at all. To use a current though somewhat sanguine phrase, they are, from the tensor analyst's point of view, monsters. But never mind the name. The important thing is that

* It strikes me that this and similar transformation rules can, with a good economy of the x's (writing or print), without ambiguity be written much more briefly and transparently thus:

$$A'^{\lambda}_{\iota\kappa} = \begin{pmatrix} \alpha\beta\lambda' \\ \iota'\kappa'\gamma \end{pmatrix} A^{\gamma}_{\alpha\beta},$$

showing the placing of the indices at the x's at a glance.

an equation written in terms of such non-tensorial concoctions would *not*, while one composed entirely of tensors (viz. $T = S$) does always retain its validity when we pass from one to another system of co-ordinates (giving $T' = S'$, that is). That this is for the physicist, or astronomer, nay philosopher, of the greatest importance need scarcely be insisted upon by many words. (General covariance of natural laws.)

A powerful means of building up [or rather 'down'] new tensors from given ones is the so-called *contraction*, an operation of wellnigh magical efficiency especially in the hands of a skilled contractor who has spent some time in the handling of this precious tensorial material. To 'contract' a tensor is to equate any pair of its indices, *one upper and one lower*, and to sum over all values (four, in our case) of that index. Thus $C^{\lambda}_{\kappa\iota}$ gives rise to

$$C^{\iota}_{\iota\kappa} = C^{1}_{1\kappa} + C^{2}_{2\kappa} + C^{3}_{3\kappa} + C^{4}_{4\kappa} = D_{\kappa} \text{ (say)},$$

a covariant first-rank tensor. By its very definition this process can be applied only to *mixed* tensors. In other words, pure, non-mixed tensors are *not* contractible, and if you do formally contract them they yield monsters. Thus, e.g., $B_{\kappa\kappa} = B_{11} + \ldots + B_{44}$ is not a scalar, i.e. not a zero-rank tensor. Nor has $C_{\kappa\kappa\lambda}$, derived from a tensor $C_{\iota\kappa\lambda}$, tensorial character. These and similar structures do not deserve a moment's attention.

In the case of higher-rank tensors we may have

two or more contractions. Thus a five-rank tensor $A^{\mu\nu}_{\iota\kappa\lambda}$ gives rise, after one contraction, to $A^{\iota\nu}_{\iota\kappa\lambda} = B^{\nu}_{\kappa\lambda}$ (say), a third-rank tensor, and this can in its turn be contracted to $B^{\kappa}_{\kappa\lambda} = C_{\lambda}$, a covariant first-rank tensor or vector. If a tensor is, so to speak, *half-bred*, that is, having as many covariant as contravariant indices, it can ultimately be boiled down to a *single* magnitude (zero-rank tensor), retaining its value in all coordinate systems; in fine, an *invariant*. Thus the tensor A^{κ}_{ι} has the invariant $A^{\iota}_{\iota} = A$ (say), such that $A' = A$. On the other hand, a *purely* covariant or contravariant tensor or a mixed one with preponderance of covariancy over contravariancy, or vice versa, has *no invariant* of its own (i.e. when unaided by other tensors).

To complete these rudiments of tensor algebra it will be enough to say a few words about the *multiplication* of tensors.

Consider any two tensors: say, both covariant vectors, A_{κ} and B_{κ}. Then the $4 \times 4 = 16$ products of their several constituents (components), $A_{\iota} B_{\kappa}$ can readily be proved to form a tensor, covariant of rank two,

$$A_{\iota} B_{\kappa} = C_{\iota\kappa}.$$

This is called the (*outer*) *product* of the tensors A_{κ}, B_{κ}, and the process of obtaining it is termed (outer) multiplication. Similarly for tensor factors of any rank and kind. Thus

$$A_{\iota} B_{\kappa\lambda} = C_{\iota\kappa\lambda}, \quad A_{\iota} B^{\kappa\lambda} = C^{\kappa\lambda}_{\iota}$$

and so on, the nature and rank of the product being in each case unambiguously indicated by the position of the indices.

The *inner product* is the outer product just defined contracted, once or more, with respect to pairs consisting each of a lower and an upper index. Thus $A_\iota B^\kappa$ gives rise to

$$A_\kappa B^\kappa = C^\kappa_\kappa \equiv C^1_1 + \ldots + C^4_4 = C, \text{ say,}$$

an invariant. Similarly $A_\iota B^{\kappa\lambda}$ leads to the important products

$$A_\iota B^{\iota\lambda} = C^\lambda$$

and

$$A_\iota B^{\kappa\iota} = D^\kappa, \text{ say,}$$

both contravariant vectors (and differing from each other unless $B^{\iota\lambda}$ happens to be symmetrical). Again, $A_{\iota\kappa} B^{\lambda\mu}$ gives rise to the inner product $A_{\iota\kappa} B^{\iota\mu} = C^\mu_\kappa$, a mixed second-rank tensor, and this in its turn yields $C^\kappa_\kappa = C$, say, a zero-rank tensor or invariant, and so on. The concept of tensor multiplication and its handling offers no difficulty whatever, and does not call for any further explanations.

We will now turn, therefore, to the differentiation of tensors (with respect to x_1, &c.), especially with a view to new tensors which may by such a process be generated or derived from given tensors, while we shall yet keep always aloof from any metrical concepts. (In fine, we are still lingering in a non-metrical manifold, an amorphous spacetime. We will metricize it presently.)

If f be a scalar or invariant function of $x_1, \ldots x_4$ or what is commonly called a *scalar field*,

$$A_\iota = \frac{\partial f}{\partial x_\iota} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

is, as we already know, a covariant tensor of rank one, or a *vector field*, the *gradient* of f, written grad f.

Other tensors, however, are not easily found by differentiation. Thus, $\partial^2 f/\partial x_\iota \, \partial x_\kappa$ is by no means a tensor. More generally, if T be any tensor of the first or any higher rank, $\partial T/\partial x_\iota$ is not a tensor; for, although this is the limit of the difference of two tensors,

$$T(x + \Delta x) - T(x),$$

divided by Δx, yet these two tensors being taken at different worldpoints, where the coefficients $\partial x_\iota/\partial x'_\kappa$ may have different values, their difference need not transform as a whole, but is disjointed, as it were, and has therefore no tensor character. What, however, is a tensor is the antisymmetric combination

$$\frac{\partial A_\iota}{\partial x_\kappa} - \frac{\partial A_\kappa}{\partial x_\iota} = B_{\iota\kappa} \text{ (say)} \quad . \quad . \quad . \quad (9)$$

which is called *the rotation* of the vector field A_ι. This consists, obviously, of but six independent components and is therefore sometimes referred to as a six-vector. A notable example is the 'rotation' (briefly 'rot') of the four-potential* which represents

* A four-dimensional union of Maxwell's vector-potential and the (scalar) electrostatic potential of a field.

the electromagnetic six-vector, comprising the three components of the electric and as many of the magnetic field of force. One more tensor derivable from a given tensor by differentiation is, for any skew tensor, $A_{\iota\kappa} = -A_{\kappa\iota}$,

$$\frac{\partial A_{\iota\kappa}}{\partial x_\lambda} + \frac{\partial A_{\kappa\lambda}}{\partial x_\iota} + \frac{\partial A_{\lambda\iota}}{\partial x_\kappa} = C_{\iota\kappa\lambda}, \quad . \quad . \quad (10)$$

which is called *the expansion* of $A_{\iota\kappa}$.

I do not know of any other tensors thus derivable, i.e. by differentiation unaided by metrics, and I doubt whether they exist. So far as my knowledge goes, nobody has ever tried to prove that the gradient of a scalar field, the rotation of a vector field, and the expansion of a skew tensor field of rank two, (8), (9), and (10) respectively, do indeed exhaust the class of tensor fields thus derivable. Perhaps some of my readers will take up this interesting question.

It is time, however, to proceed with the subject and to metricize our fourfold $x_1\, x_2\, x_3\, x_4$, or to endow the hitherto amorphous world, the spacetime, with *metrical properties*. This business is, especially for the physicist, the astronomer, and the cosmologist, of such vital importance that it will be well to treat it at some length and with the greatest possible care, the more so, as usually the conceptual side of the process of metricizing our world is treated with a good deal of obscurity and, even in the case of some prominent relativists, with a sprinkle of a very undesirable mysticism.

A good approach to this question can be made with the aid of ordinary, three-dimensional, space as an analogy. Let us therefore forget for the present the fourth variable (x_4), the time, that is, and let us think of space ($x_1\, x_2\, x_3$) itself. In other words, let us imagine the world *at a given instant* of time; in figurative language: a section of the world. Let this three-space be, at first, quite amorphous, i.e. just a threefold continuum of points. Any triple system of surfaces, each bearing three numbers, taken merely as labels, viz. x_1, x_2 variable, x_3 constant; x_2, x_3 variable, x_1 constant, &c., can be imagined drawn so as to divide the space into cells, each of these limited by six walls, as it were, elements of three pairs of those surfaces. Such is the meaning of the variables x_1, x_2, x_3 appearing in all our preceding considerations. They are what are called general or, more commonly, *Gaussian coordinates.* None of them has anything to do with lengths, or distances, or angles as measurable quantities. A triple of labels x_1, x_2, x_3 fixes a point or a place (relative to some framework), another triple characterizes another place, and so on, the only essential requirement in choosing these coordinates or numerical labels being that to every two *distinct* points should correspond two *different triples of labels.* The converse need by no means be true. That is to say, a given point may have, say, a unique label x_1 but an infinity of x_2 and x_3 labels, as e.g. $x_2 = b$, $b+2\pi$, $b+4\pi$, &c., and $x_3 = c$,

$c+\pi$, $c+2\pi$, $c+3\pi$, &c. (Just think of the common polar coordinates, forgetting, however, their metrical significance.)

In the space thus conceived we may have various topological properties, where such concepts as 'inside' and 'outside', 'one- or two-sidedness' of a surface, 'simple or multiple connectivity' of a region play a role, but where shape and size of lines and segments, surfaces, or solids find no place. This will be the purely qualitative space, as it were, of Analysis Situs.

Nothing more can profitably be said, for our present purpose, that is,* about the space thus conceived.

In the next stage let us enrich somewhat this concept by introducing the concept of '*straight lines*' or, briefly, *straights*. Of course, we are not going to define them (as was the case in older text-books which speak of these lines as 'lying evenly', being 'the shortest paths' between any two of their points, and what not), but simply accept 'straight' as a primitive or undefined concept, although my readers will find it hard (at first) to divest it from the familiar, visual or tactual image, appearance, or 'feel' of such a line. Naturally the 'straight' thus introduced would be a mere word and thus utterly

* Theoretically, abstractedly, a good deal more could, of course, be said about the amorphous space. Nay, some modern mathematicians have found it possible and profitable to fill out quite bulky volumes with Analysis-situs topics and their hair-splitting scrutiny.

barren of consequences or implications, unless we enounce about it some 'axioms' or, less obscurely, some *unproved propositions*. (Every mathematical science is, logically, erected upon a set of undefined concepts and unproved propositions. Arithmetic, for instance, is no better, and no worse, in this respect than geometry.) Thus, we will say: 'A pair of distinct points A, B lies, in general, on one straight only', or 'A straight is uniquely determined by any two of its points', with a possible exception of some pairs (antipodes). And so forth, as the reader will find in any modern list of axioms of what is often called geometry of position. A mere handful (five or six) of such unproved propositions about the 'straight', with the obvious definition of the 'plane' as generated by straights, is sufficient to erect the whole of that marvellous, rich, and unspeakably beautiful monument, *Projective Geometry*, the science* of the *projective space*.

It would, naturally, not answer the main purpose of this book, and occupy too many pages, to expound here all, or only all prominent, properties of projective space. It will be enough to state some of these beautiful properties, especially such as are likely to pave the way (if only by analogy) to what is to follow.

* Started by the great French geometers and masterly completed by von Staudt. The reader not fully conversant with the subject will find much solace and instruction in Halsted's condensed and Young and Veblen's masterly developed book on Projective Geometry.

Let any three points, marked with the labels 0, 1, ∞,* be chosen on a straight l (Fig. 1), and let, for the sake of the argument, our attention be concentrated entirely upon this straight, as a particular

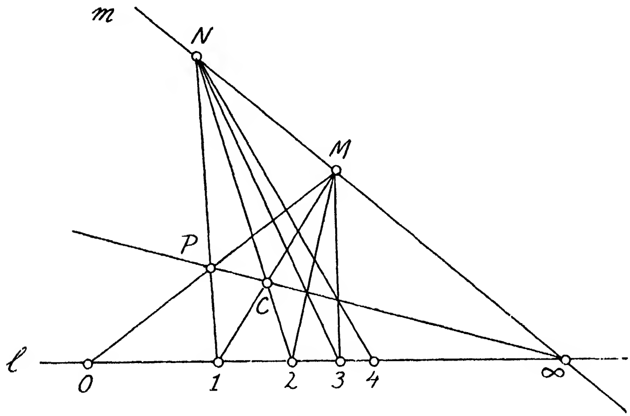

FIG. 1.

one-dimensional manifold, a sub-manifold of the projective space in hand. Then the remarkable thing is that these three points *determine uniquely* a *fourth point* on the same line, known as *the fourth harmonic* (conjugate to 0, say). To find this new point draw through ∞ any straight m, and through 0 and 1 any two straights $0M$, $1N$ intersecting in P. Next,

* Needless to say, there is, thus far at least, nothing infinite about the label ∞, nor anything nihilistic about 0, nor yet anything unitarian about 1. They are just three labels, as good as A, B, C, or α, β, γ, and chosen above with arithmetical connotations merely for future convenience.

trace the straight joins $P\infty$ and $M1$. Let these cross in C. Then the straight NC produced will find on l the required 'fourth harmonic', to which we may conveniently attach the label 2. (This is v. Staudt's construction.) It is true that this perfectly definite new point of the l-world was arrived at by an extra-mundane (extra-l) construction, viz. by drawing a straight m and a number of other auxiliary straights, all of course distinct from l. Yet the position of the new point '2' is wholly independent of these auxiliaries,* and is thus a property of the l-world itself, i.e. an intrinsic property: the three points 0, 1, ∞ on l *co-determine* a fourth point, 2, of the same straight. Is not this admirable? Just imagine you were in some desert, say Sahara. Around you all is uniform sand, and nothing by which to discriminate one 'place' from another, except three palms, just collinear, or perhaps three such columns erected by man. Let these be marked 0, 1, ∞. Then, in spite of

* That is to say, if you drew, instead of m, another straight m', and so on, you would always find the same point, 2. Of course, the process of 'drawing' is meant to be performed with the means of a common straight-edge, which is a concrete representation of the originally abstract 'straight line', and but one of an infinity of possible 'representations'. And the remarkable thing is that the theorem (uniqueness of fourth harmonic, conjugate to 0) holds, with any desirable precision (when drawn with a sharply pointed pencil, say) for this concrete representation. I cannot help confessing here that I have repeated innumerable times this multiple construction and experienced each time a genuine thrill of pleasure at finding thus the theorem 'experimentally' verified. And I can warmly recommend this instructive and stimulating pastime to the reader.

the hopeless wasteness and dreary monotony (uniformity) of that sandplane, you can give your friend an appointment in Staudtian code ('Meet you at 2') and are sure not to miss him or her—apart from time considerations, of course. For both parties can draw in the sand the requisite auxiliary lines. Your own construction may soon be obliterated by wind or mischievous intruders. This, however, will matter but little. For, whatever your friend's particular sand drawing, which is to be made all afresh, he is sure to reach the same place.

It goes without saying that the same process can be repeated (finding fourth harmonic to 1, 2, ∞, conjugate to 1), leading to yet another uniquely determined point, which we will call 3, and so on (cf. Fig. 1). The point 3 will lie beyond 2 and this side of ∞, similarly 4 will fall between 3 and ∞, and so forth, the points thus consecutively arrived at crowding towards but never reaching the terminal. In mathematical phraseology, to reach this point an infinity of Staudtian constructions, viz. passages such as from 1 to 2, from 2 to 3, &c., or briefly an infinity of Staudtian steps, are required. And that is the reason why this point was given the label ∞. It may now be said that by three given collinear points, 0, 1, ∞, an infinite (but denumerable) plurality of other points, 2, 3, 4, &c., is entirely co-determined. All these will be placed on the segment 01∞ of the line l, to be distinguished from the remaining part,

or complementary segment, of this line. To cover also this segment with points uniquely co-determined by the same triad 0, 1, ∞, it is enough to construct the fourth harmonic conjugate to 1 (instead of the previous 0). This will, in the case of our figure, fall to the left of 0. It may consistently be labelled -1. Similarly the points -2, -3, &c., will be obtained, the first few of these being placed beyond 0, and the following ones beyond (to the right of) ∞. Thus the terminal point (∞) receives also the label $-\infty$. This, however, does not, obviously, give rise to any ambiguity. Distinct points have throughout distinct labels. Nay, the correspondence of points and labels (positive and negative integers) is bi-univocal or a one-to-one correspondence, except the singular point which has two labels, ∞ and $-\infty$.

The process just described can readily be extended to points intermediate between 0 and 1, 1 and 2, &c., and similarly the negatively labelled ones. These will receive fractional labels, $\frac{1}{2}$, $\frac{1}{3}$, &c., $\frac{2}{3}$, &c. Their construction, though easy enough, will however not be described here. It can readily be looked up in a good text-book of projective geometry (as, e.g., Young and Veblen's, cited above). Suffice it to say that to every rational number, positive or negative, corresponds a point of the segment 01∞ and of the complementary segment, respectively, and that the sequence of these points is the same as that of their numerical labels. The extension to irrational num-

bers, and the corresponding points of the straight l, which is based upon the Archimedean postulate (continuity, projectively generalized), does not offer any difficulty. In fine, to every real number n corresponds a definite (unique) point of the line and vice versa, with the exception of the singular point of the Staudtian scale, which receives the two labels ∞ and $-\infty$. There is, of course, no actual (geometrical) singularity about this point; for, clearly, any point on the line l can be picked out as such a 'terminal' point, in much the same way as *any* point can be made the zero, or the unity, of the scale.

The ordered point-pair $(n) \rightarrow (n+1)$, where n is an integer or not, can appropriately be called a *projective unit step* or, as proposed elsewhere,* a *Staudtian*, and any two points A, B of the line l can be said to be a certain number of Staudtians apart, or, also, the latter may be referred to as the measure of their 'distance'. If, by any processes of projection and section, the line l is ultimately projected into some other straight l' of the projective space, the Staudtian distance of the point-pair A', B' corresponding to A, B is *the same* as for the latter pair. In brief, it is a projective invariant, and this makes it the more worthy of notice. It will be kept in mind that to make this Staudtian distance $\overline{AB}$ unambiguous, it is necessary to specify the path taken from A to B,

* L. Silberstein, *Projective Vector Algebra*, Bell & Sons, London, 1919.

e.g. by requiring that the singular point ($\pm\infty$) of the scale should not be crossed.

The same precaution, of course, is indispensable with regard to ordinary metrical distance as measured by a 'rigid' rod, unless the straight is declared to be open, i.e. stretching on both sides to 'infinitely distant' points which are not conceived as coinciding, and thus in either Euclidean or Lobatchevskyan metrical space; while in Riemannian, spherical or elliptic, antipodal or polar, the straight lines are re-entrant, their total length being finite. From the projective point of view it was a long time ago found convenient to consider even the Euclidean straight as closed, i.e. the two points at infinity as coalescent. With the Staudtian scheme such is automatically the case, the 'terminal' point (which, of course, may be any point of l) bearing both labels, $+\infty$ and $-\infty$.

Such a numbering of points can readily be extended to any plane, by choosing in this plane two arbitrary straights, l_x, l_y, taking their cross as the zero of the scales, any two points, one on each, as the unitpoints, and any two more points as the terminals, say $x=\infty$ and $y=\infty$. Finally, adding a third axis l_z non-coplanar with l_x, l_y and marking on it an arbitrary unitpoint and an equally arbitrary terminal, any point of the space can be numerically labelled. (See, for instance, Young and Veblen's *Proj. Geom.*, vol. i.) The Staudtian numbers, x, y, z thus obtained are commonly termed the (non-homogeneous) *projective coordinates* of a point of the

space. The correspondence of points and number triads is again bi-univocal, apart from the singular plane which is that passing through the three terminal points selected upon the axes, $x = \pm\infty$, $y = \pm\infty$, $z = \pm\infty$. The important thing, however, is that without making use of the common concept of metrical distance (implying measurement by 'rigid' rods, or 'inextensible' chains, or what not) the points of space can be numerically labelled in such a way that the knowledge of only the origin (zero), the terminal (singular) plane, and the three unitpoints enables us to arrive without failure, with the aid of a straight-edge, to the point of space designated to us by any (real) number triple x, y, z.

Before leaving the projective space, let us still recall certain features of *conic* curves and *quadric* surfaces. Both of these, or at least any desired number of points lying on a conic or a quadric, can be constructed projectively, i.e. with the aid of a straight-edge (and a 'plane-layer') only.

Both of these constructions can be read up in any text-book of projective geometry. Yet it will be useful to give here at some length a projective construction at least of the *conics*, viz. point-conics, in a given projective plane. This, given some time ago in a little book (*Projective Vector Algebra*), differs in its procedure from the usual construction, but is, of course, essentially equivalent to it. Our method of generating the point-conic has, perhaps, the addi-

tional advantage of being expressible algebraically in a remarkably simple form.

Let O, A, B be three (of the *five* freely prescribable) points of a conic and P a fixed point, say outside the triangle OAB, as in Fig. 2. Through P draw any straight t_1. Let this cross OA, OB in T_A and T_B. Then the cross $AT_B \, . \, BT_A$, marked in the figure by 1,

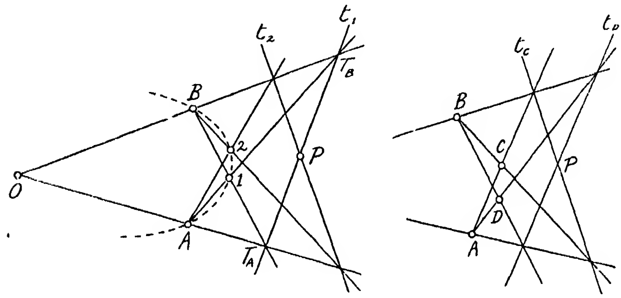

Fig. 2.

will be a point of the required conic. If the vectors or ordered point-pairs $\overrightarrow{OA}$, $\overrightarrow{OB}$, are denoted by **A**, **B**, then the vector $\overrightarrow{01}$ will, with t_1 as line of termini, be the projective *sum* of **A** and **B**, which may be written

$$\overrightarrow{01} = (\mathbf{A} + \mathbf{B})_{t1}.$$

Similarly (cf. Fig. 2), $\overrightarrow{02} = (\mathbf{A} + \mathbf{B})_{t2}$, and so on. Generally, for any t-line passing through P we may write for the vector sum **R** of **A** and **B** taken with respect to this line,

$$\mathbf{R} = (\mathbf{A} + \mathbf{B})_{t(P)}.$$

The endpoints of the generic vector **R**, always with

O as origin, are readily proved to lie all on a conic.* This, then, is the equation of a conic,† passing obviously through O, A, B. Nay, it can be shown that *every* conic can thus be constructed. In fact, it is well known that any conic is determined by *five*, and just five, arbitrarily chosen points. Now, three of these are O, A, B, which can be selected at our pleasure. Let C and D be the remaining two points, again chosen at will. Then also $\overrightarrow{OC} = (\mathbf{A} + \mathbf{B})_{t_C}$, $\overrightarrow{OD} = (\mathbf{A} + \mathbf{B})_{t_D}$, where t_C, t_D are the positions of the t-line corresponding to these points. Whence follows easily the construction of the centre P of the pencil of t-lines, namely as the cross of t_C, t_D, and this construction is shown in Fig. 2. This, however, proves the statement: Every conic can be written $\mathbf{R} = (\mathbf{A} + \mathbf{B})_{t(P)}$. Needless to say, the projection of a conic upon another plane and, say, its re-projection on the original one is again a conic.

The otherwise familiar distinction between ellipse, parabola, and hyperbola (including a pair of straights) is, from our present projective point of view, entirely irrelevant, nay meaningless, in spite of the apparently striking difference between 'open' (infinite) and 'closed' (finite) curves. For none of these properties is projectively invariant; a hyperbola or

* Whose projective definition runs thus:

If two coplanar flat pencils are projective but not perspective, the crosses of correlated straights form a *conic* range.

† A *point*-conic, that is, since it is generated by points, crosses of pairs of lines.

parabola may be changed by projection into an ellipse, and vice versa. Still less can one 'ellipse' be distinguished from another (the 'lengths of axes' and the 'eccentricity' being metrical concepts, foreign to our present circle of ideas. Nor is there any significant sense in speaking of a 'circle' as an ellipse of a particular shape (zero 'eccentricity').

In fact, a *circle* is defined as the locus of points '*equidistant*' from a given point O, called the centre of the circle. Thus, if P be any point of the circle and if we write OP for its distance from O, the equation of the circle is

$$OP = \text{const.}$$

This 'constancy of distance', however, is meant to be something which is stated in terms of 'rigidity',* and this, of course, is physics (not a pure space-theory), implying constancy of temperature of the material generating this curve, and possibly a host of other conditions under which the actual construction has to be performed.

But these remarks need scarcely be dwelled upon. For, have we not stipulated insistently that no instruments other than a *straight-edge* are to be used in this, the projective, stage of our thinking of space?

We might invoke the aid of Staudtian steps and distances which, as we just saw, are certainly

* Say, a sharp point (pencil) fixed in a bar of steel describing the 'circle' in question, while the bar is turned about some other point O fixed in the bar and on the drawing-paper.

independent of the use of rigid bodies. Quite so. But if P_1, P_2 be two distinct points (not antipodal) of the conic in question, the choice of a unit step or Staudtian upon the line OP_2 is entirely independent

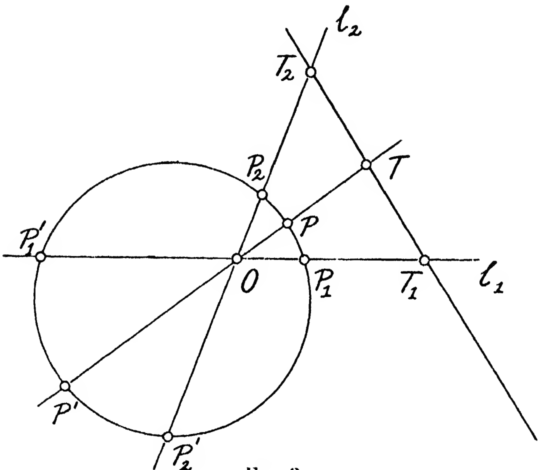

FIG. 3.

of that upon OP_1, so that there is no definite sense in 'comparing' the numbers of Staudtians contained in OP_1 and OP_2. For this to become definite a further *convention* is necessary. In fact, we can draw around O *any curve* (not self-intersecting) and, if P_1, P_2, &c., be points upon it, declare OP_1, OP_2, &c., to be *unit steps* or Staudtians on the straight lines l_1, l_2, and so on (Fig. 3). But, to be at all able to speak of such steps, of their multiples, sub-multiples, &c., we must introduce some (arbitrary) straight as line

of termini. Well, let this be the straight t. This being, once and for all, selected, the nature of the curve which is to play the role of a gauge or standard curve will no longer be arbitrary. In fact, if P_1' be the antipode of P_1, i.e. the curvepoint collinear with O, P_1, and similarly for P_2, &c., the Staudtian number to be attached to P_1' will be -1. In other words, T_1 being the terminus of l_1 ($T_1 = OP_1 . t$), P_1', O, P_1, T_1 will be a harmonic range with O, T_1 and P_1', P_1 as conjugate pairs, and the same thing will be true of the four points P', O, P, T corresponding to any straight passing through O. But this property is well known to belong to a *conic*, and no other curve, namely, one whose centre (O) is the *pole* of the line t.

Thus the desired gauging curve must be a conic and the corresponding t-line must be *the polar* of its centre O, supposed to be given at the outset.

The standard or unit conic ($r = 1$) being thus fixed, all homologous and concentric conics

$$r = n = \text{const.},$$

where n is any positive number (number of Staudtians contained in semi-diameter), can at once be constructed, point by point. Thus, the label of P_1 being 1, it is enough to construct on OPT the point having the label n on the Staudtian scale, and similarly for any other point of the standard conic. Again, abandoning O as the origin of all vectors, if O' be any other point of the plane, a vector $\overrightarrow{O'P'}$

equal to OP can be drawn by a simple construction (cf. *Proj. Vect. Algebra*), the definition of equality implying that the lines of the two vectors intersect on the t-line, and that the passage from O' to P' is again accomplished by a single (unit) Staudtian step. Thus also the number of Staudtians contained in any segment AB whatever or the projective 'distance' of any two points A, B of the plane can at once be evaluated. A number of other implications of the introduction of a standard conic (which may well be called a 'circle') will be found in a sequel to my *Projective Vector Algebra*.* For our present purposes the preceding considerations will suffice.

Before leaving the projective plane (and space) one more interesting application of the Staudtian scale, of points, that is, labelled by positive *and* negative numbers, may here be mentioned. If O, U, T be the initial, the unit, and the terminal points on a straight l, then, as we already know, each point on the segment OUT has a definite positive numerical label n, and vice versa, to every positive n corresponds one and only one point of this segment. The points having *negative* Staudtian numbers for their labels lie all on the complementary segment of the line l, beyond O and beyond T. Now, all the axioms of projective geometry being supposed to hold, three different cases present themselves as to the intersectional pro-

* 'Further Contributions to Non-Metrical Vector Algebra', *Phil. Mag.*, vol. 38, July 1919.

perties of coplanar straights. Let l be a straight, P an external point, and $s = s(P)$ a plane pencil of straights with centre P and coplanar with l. Then the three possible cases are: first, a part of the pencil of s-lines do not intersect l in *actual* points,* and this class of non-intersecting lines is divided from that of intersecting ones by *two distinct straights* (Lobatchevskyan

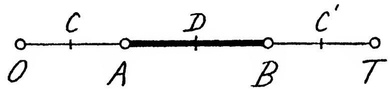

Fig. 4.

parallels), or second, there is but one non-intersecting line (Euclidean parallel †), or third, every line s has an actual intersection point with l (no parallels). The first case is that of the *Lobatchevskyan*, the second of the *Euclidean*, and the third of the *Riemannian* plane, with which usually also the names hyperbolic, parabolic, and elliptic (or spherical) are associated. Now, $x = n$ (Staudtians) being, for n positive, a point X of the segment OUT, the point $x = -n$ is the fourth harmonic to OXT, conjugate to X, and, taking account of its well-known construction, this fourth harmonic does not exist as an actual point, if X falls within a certain segment AB, contained within OT (Fig. 4), and does exist, if X

* Though they always have in common with l what is called an 'ideal point', i. e. determining a definite plane pencil of lines. (Cf. *Proj. Vect. Algebra*, p. 34.)

† It goes without saying that these parallels have not the property of projective invariance.

lies in the remaining part of OT. Thus, e.g., C or C' have, while D has not, an actual fourth harmonic. Now, in hyperbolic space A and B are *distinct* points, in parabolic (Euclidean) they coalesce, forming a single point, and in elliptic space there are no such points at all, the whole segment OT consisting of points having an actual fourth harmonic. In common phraseology these three cases may be characterized by saying that A and B, when distinct, are the points conjugate to either of the *two* 'points at infinity' of the Lobatchevskyan straight, that when they coalesce into one point, this is the conjugate of the unique point at infinity of the Euclidean straight, while there is no 'point at infinity' on the Riemannian straight which is actually re-entrant and, in the usual, metrical sense of the word, of finite total length.

After this lengthy but, perhaps, not superfluous digression, let us return to our four-dimensional world or spacetime. This is, thus far, quite amorphous. Let us now inquire into the procedure of metricizing it and into the physical significance of this procedure.

Let us, first, try to do it by imitating projective three-space geometry. As a matter of fact the extension from three- to four- or more-dimensional spaces offers no difficulty and has been actually accomplished by projective geometers years ago. (See, e.g., Young and Veblen's vol. ii, where sets of axioms for

polydimensional projective spaces are given and treated with much care.) Yet, since in our case one of the 'dimensions' (that which later on will be associated with x_4, say) is, psychologically at least, of a very different nature from the remaining three, it will be better not to rush over this new situation by spatial (visual or tactile) imagery and corresponding language, geometrical, that is, but to be more explicit and abstain, at first at least, from metaphorical language. Let us, in fine, look the situation straight in the eyes, so to speak.

A *particle* of matter, relatively small enough to be confounded with a *material point*, as it is called by continental writers, occupies at a certain instant a a point or 'place' A, and at some later instant b is found to be at another point B. (The letters a, b do not at this stage, of course, denote magnitudes, 'time intervals', elapsed from a certain origin of time reckoning, but just mere labels of time instants, exactly as A, B have nothing to do with distances or angles but are labels of places related to some frame of reference.) Then A, a is a worldpoint; call it α; and $\beta = B$, b is another worldpoint. There is clearly no sense in asking how 'distant' these two worldpoints are from each other, or what is the 'interval' separating them. To be able to attach a meaning to such a question the world has first to be metricized.

Between α and β there are other worldpoints be-

longing to the particle which can be ascertained by observation. We may say at once, having in mind a moving particle, that from α to β stretches a whole continuity of worldpoints or a continuous piece of the particle's worldline. The (geometrical) path of the particle, stretching from A to B, does not, of course,

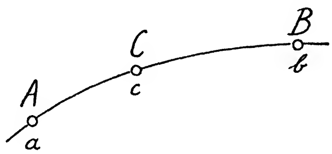

FIG. 5.

characterize by itself the worldline. To complete it one would have, e.g., to attach time labels, marking individual instants, viz. affixing a to A, b to B, and c to some intermediate point C of the path, and so on (Fig. 5).

In fact, such a procedure is applied in certain railway time-tables, where a, b, c assume some such shape as 1^{00} p.m., 1^{30}, 2^{00}, with, however, the additional implication that the time intervals 1 to 1^{30} and 1^{30} to 2^{00} are 'equal', which in the present stage of our considerations is, of course, superabundant.

The task of placing such time marks all along the path would, naturally, soon become awkward, especially if we wished to insert between any two instants other instants, and so on. And the scheme would entirely collapse if we claimed to thus represent the whole continuous history of the particle, in

fine, its full (not dotted) *worldline*. To remedy this the relativists commonly imagine (or *say* that they imagine) a 'fourth dimension', a fourth axis, the time-axis, as they say, along which a continuous succession of time labels is supposed to be placed. Then, the particle's worldline *is* a line, some curve, in that four-dimensional space. But these, of course, are mere words. Geometrical language applied to partly non-spatial relations is, no doubt, helpful in many cases, yet it is at the same time often misleading, and before we succumb to the general trend and plunge into its use (if only for the sake of brevity), it is better to abstain from it, and from the corresponding diagrams (e.g. with but one space-axis for abscissae, and time-axis for the ordinates of worldpoints). A good proposal in this direction has been made by Timmerding some fifteen years ago. This, although implying the metrical notion of spheres and the size of their radii, is worthy of notice and may profitably be given here.

Let us then imagine, after Timmerding, that our particle is luminous and that it emits spherical waves, but in a rather artful way, viz. so that it always remains the centre of the wave originally emitted (starting at A, say) and continuously expanding.* In this way the points of the path AB will all

* This would, in part, harmonize with the 'ballistic' behaviour of light emission (defended of late by De La Rosa), which, however, is in clashing discordance with observed facts.

be surrounded by spheres of increasing radii, centred at these points, and the worldline of the particle will be represented by the envelope of all these spheres, an ever broader and broader tubular surface with the path as axis, which in general will be a tortuous curve. This is not a bad representation of a worldline, but not much is gained by it in further

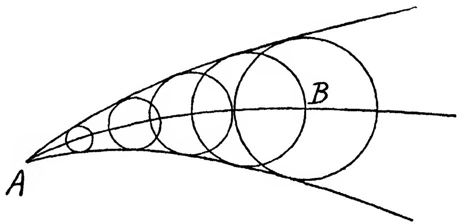

FIG. 6.

developments relating to the physical spacetime or actual world. We will, therefore, not dwell upon this or any other graphic representation any longer. In order to imitate, in spite of these difficulties of visualization, the scheme which has led to projective space theory, we may proceed as follows.

Let us introduce, as an undefined concept, that of *free motion* or, in the limit, *light propagation*, and, just to have a distinct name, call the corresponding worldline a *natural worldline.* (Later on, when our world is metricized, this name will be replaced by 'geodesic'.) This is to play in the world a role analogous to the straight line of projective geometry, spacepoints being now replaced by worldpoints,

such as $\alpha = A, a$; $\beta = B, b$. If we now enounce about the natural worldlines the same unproved propositions (axioms) as were formulated in projective geometry, our amorphous world will become what may be called a *projective world*. If no further clauses were introduced, this would not be distinguishable from a four-dimensional projective space, and would scarcely serve any physical purpose. The supplementary, distinctive features are not hard to perceive. In fact, while every pair of spacepoints is (projectively) as good as any other, this cannot, in view of certain physical facts, be claimed for pairs of worldpoints. To be more explicit, while a straight can be drawn through any two points of space, a natural worldline cannot be laid through every pair (α; $\beta = A, a$; B, b) of worldpoints.

Let, at the instant a, a *light* signal be started at a point A, and let it reach the point B at some definite later instant b^*. (The worldline, 'lightline', stretching from A, a to B, b^* is, by assumption, a natural one.) Then, if

$$b \,(\, b^* \text{ (read: } b \textit{ earlier than } b^*\text{)},$$

we will declare that through the worldpoints A, a and B, b *no* natural worldline can be drawn. This is equivalent to saying that no particle in free motion* can overtake light. But if $b \,)\, b^*$, a natural line (and one only) can always be laid through A, a and B, b.

* And, as may readily be deduced, no particle at all, i. e. moving in *any* physically admissible manner.

These 'natural worldlines', any one of which may actually be the worldline of a particle, we consider, abstractedly, as a sub-class of what we may provisionally call four-dimensional straights or, more briefly, *four-straights*, and of these we may require unrestrictedly to obey all the projective axioms.

Thus, a four-straight, and only one, will pass through *every* pair of worldpoints, but only *some* four-straights will be natural worldlines, as explained above.

Rest being but a special case of free motion, the worldline of a particle at rest will be a four-straight, the lower limit—as it were—of all natural worldlines. This can be spoken of as the *time axis*. A triad of other four-straights, *not* natural worldlines, can be chosen as *space axes* (these will also be ordinary or three-space straights). All fundamental theorems of projective geometry being valid for this projective world, Staudtian scales and the corresponding projective coordinates x_1, x_2, x_3, and x_4, to be reckoned along these three space axes and the time axis just mentioned, can at once be introduced. This coordinate system will then be a special, concrete, case of the Gaussian coordinates mentioned at the outset.

In this way, without yet introducing metrics proper, i.e. without ever appealing to 'rigid' bodies or to any quantitative feature of light propagation, we can characterize any worldpoint by four real numbers, $x_1\, x_2\, x_3\, x_4$, which (the origin, the unit, and

the terminal points being chosen) have a perfectly definite meaning.*

One might think, however, of a possible objection, namely, that such a procedure is perhaps not general enough for the physical world, that is, for that space-time with which the physicist and the astronomer has actually to deal. In fact, a *projective space* has been proved by Schur † to be equivalent to a (metrical) space of (isotropic and therefore also) *constant curvature*, positive, zero, or negative. This means that a manifold in which the whole set of axioms of projective geometry holds can have only such metrics impressed upon it, as converts it into a manifold of isotropic and herewith homogeneous or, briefly, *constant Riemannian curvature.*‡

Now such a manifold is certainly too narrow to cover the actual world of ours, the physical space-time, filled as it is, here and there at least, with matter, whether of the palpable kind, consisting of atoms or only naked protons or free electrons, or of radiant, electromagnetic energy. For this spacetime is, demonstrably, not homogeneous as to its curvature properties, very much as the surface of an egg or, better, that of an orange, roughly spherical but full of corrugations, dimples, and warts. This world,

* For details see my paper in *Phil. Mag.*, vol. 50, 1925, p. 681.

† F. Schur, *Mathematische Annalen*, vol. 27, 1886, p. 537.

‡ For a brief explanation of this fundamental concept of differential geometry, see art. 'Tensor Analysis' in the new edition of the *Encyclopædia Britannica.*

then, cannot be faithfully covered by a projective spacetime. Yet, if we either imagine for the moment that all matter has been removed, or are content to consider regions far away from the more conspicuous lumps of matter (planets and stars), the projective manifold can render us good services. In fine, we may accept such a spacetime as representing the actual world *on the whole* or, if the reader so wills, the world *at large*.

This amounts, in plain English, to assuming that our world, spacetime, as the arena of events, has, on the whole, everywhere and always the same properties and, therefore also, the same curvature, *mean curvature*, as it is technically called, and accordingly, the same *curvature radius* $\mathfrak{R}$, say.

This seems, so far at least as our present knowledge goes, quite plausible. I mean that we do not, thus far, have any serious reason to suspect that there is or was somewhere in intergalactic regions a place or a region which within the limits of some long time-interval behaves in a way significantly distinct from other places and instants, or portions of spacetime.

It may be well to point out that the much-extolled *relativity* of position (place, and time-date as well) holds good only in such a world, a *homogeneous* one on the whole, that is. If it were not so, we might have *absolute* positions and instants (dates) in very much the same way as there is, in a perfectly

significant sense of the word, absolute position on the surface of a bird's egg or, say, a three-axial ellipsoid (with its greatest curvature on the ends of the longest axis, &c.), while there is none upon a spherical surface, each considered as the abode of some intelligent bi-dimensional beings. This obvious state of things was, to my knowledge, first pointed out by Clifford.*

Let us then contemplate our spacetime as a fourfold of *constant curvature* (K), having throughout the same *curvature radius* $\mathfrak{R}$, that is,

$$\mathfrak{R} = \sqrt{K^{-1}}, \text{ if } K > 0,$$

or

$$\mathfrak{R} = \sqrt{-K^{-1}}, \text{ if } K < 0,$$

and, of course, $\mathfrak{R} = \infty$, if $K = 0$.

Whether this world-curvature K is at all sensibly different from zero, and if so, how big it is (in positive or negative reciprocal acres or square miles, or square parsecs), is not a question to be decided by the mathematician alone, sitting at his writing-desk, but by (him jointly with) the observing astronomer, and to a certain extent, also the experimenting physicist.

It is exactly the purpose of this book to find out whether there are reasons weighty enough to discriminate between $K \gtreqless 0$ and to place the value of the world-curvature between certain numerical limits.

* W. K. Clifford, *The Common Sense of the Exact Sciences*, 5th ed., London, 1907, p. 222.

It is this problem which in the title is briefly referred to as the determination of the *Size of the Universe*.

In approaching it, our next task will now be to put the projectivity or, equivalently, the constancy of the curvature of our spacetime on the whole, into a form indicated by the use of those generally co-variant mathematical entities, the tensors, which were explained in the preceding pages.

Imagine a (three-dimensional) pencil of infinitesimal, co-initial vectors or point-pairs, say with O (x) as common origin and P $(x+dx)$ as endpoints, where x is written, summarily, for x_1, x_2, x_3, x_4. As already stated, the first-rank contravariant tensor or vector dx_ι $(\iota = 1, 2, 3, 4)$ has no invariant of his own, no 'size' or 'length', and this continues to be true even if we agree to consider it as the (directed) element of a projective straight line in a projective four-space. The various vectors issuing from O, say $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$, &c., have nothing (but the origin) in common with each other. We may say of them only that they differ in 'direction', while there is nothing like 'size' or 'length', with respect to which they could be compared with each other as to equality or non-equality.

Let us, using a figurative language, imagine the worldpoint O surrounded by an infinitesimal projectively constructed quadric, of course a three- not two-dimensional locus of worldpoints, as standard hypersurface, very much as in the case of a projec-

tive three-space.* The equation of such a hypersurface, very much as that of a conic, in x_1, &c., considered as projective coordinates, say, is of the second degree, and, if g_{11}, g_{12}, &c., be some functions of the coordinates x of the origin alone, it can be written

$$g_{11}\,dx_1^2 + 2g_{12}\,dx_1\,dx_2 + \ldots + g_{44}\,dx_1^2 = \text{const.},$$

or briefly, using the previously explained summation convention,

$$g_{\iota\kappa}\,dx_\iota\,dx_\kappa = \text{const.} = ds^2, \text{ say.}$$

All infinitesimal vectors dx_ι drawn from O to this hypersurface will now be declared equal to each other as to *size* or *length*, their squared size or *norm* being denoted by ds^2. This, unlike ordinary geometry, need not necessarily be positive, but may as well vanish or have a negative value. If it vanishes, each of the corresponding vectors (all vectors of the pencil under consideration) is an element of a *singular* or a *light line*, representing light propagation, that is; and if positive (with certain clauses as to the sign of g_{44}, &c.), an element of a possible world-line of a particle or a *time-like* vector; finally, if negative, a *space-like* vector.

The coefficients $g_{\iota\kappa}$ themselves, clearly subject to the condition $g_{\iota\kappa} = g_{\kappa\iota}$, may, thus far, be any functions of the coordinates, in general that is, ten

* For details of such a treatment concerning, explicitly, space-time itself, see my paper quoted on p. 36.

mutually independent functions of all the four x. Yet, even at this stage a certain limitation of these ten coefficients can be noted. In fact, we will naturally claim for the equation of this gauging hypersurface the general invariance, i.e. independence of the choice of coordinates in which it is written down. In other words, the quadratic form

$$ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa \quad . \quad . \quad . \quad . \quad (10)$$

is to be a general invariant, a scalar or zero-rank tensor.

Now, dx_ι being a contravariant first-rank tensor, this requirement can be satisfied only if $g_{\iota\kappa}$ is a covariant second-rank tensor. In fact, it can readily be proved, that if the inner product $A_{\iota\kappa}B^\iota B^\kappa$ is a scalar for *any* contravariant vector B^ι, $A_{\iota\kappa}$ must be a tensor, covariant of rank two. And such, exactly, is the situation in our case.

Thus $g_{\iota\kappa}$ in the quadratic form (10), called also most commonly the (squared) *line-element*, is a symmetrical covariant tensor of rank two.

We can invert the order of this reasoning and, instead of claiming at the outset the invariance of that, i.e. of some quadratic form which is to represent the size of the vector dx_ι or the *distance* or *interval* between its origin and endpoint, we may say that a certain symmetrical second-rank tensor $g_{\iota\kappa}$ is being *impressed* upon the spacetime as a 'fundamental' or *metrical* tensor (field), converting it into a metrical world, very much like we speak of an electromagnetic

six-vector impressed on a space and converting it into an electromagnetic field. Then $g_{\iota\kappa} dx_\iota dx_\kappa$ will be an invariant.

At the same time, of course, all other contravariant vectors, say A^ι, which hitherto had no invariant, will, with the aid of $g_{\iota\kappa}$, acquire one, viz. $g_{\iota\kappa} A^\iota A^\kappa$, the norm or squared size of A^ι. Similarly, any symmetrical second-rank tensor $B^{\iota\kappa}$ will now have the metrical invariant $g_{\iota\kappa} B^{\iota\kappa}$, the inner product (*twice* contracted) of the two tensors.* Again, any tensor of second or higher rank will lead to other tensors metrically associated with it, as e.g. $g_{\iota\kappa} B^{\iota\lambda} = B^\lambda_\kappa$, a mixed tensor associated with the purely contravariant $B^{\iota\lambda}$. Similarly $g_{\iota\kappa} B^\kappa = B_\iota$ is a covariant vector, the conjugate of the contravariant vector B^ι, and so on.

In what follows the determinant of the $g_{\iota\kappa}$ will be denoted by g, and the minors of g divided by g itself will be written $g^{\iota\kappa}$. These, as can readily be proved, form again a tensor, contravariant, of rank two and, of course, again symmetrical. This may be called the associated metrical tensor. It goes without saying that $g^{\iota\kappa}$ fixes the metrics of the world quite as well as $g_{\iota\kappa}$. The associated tensor can again be used to derive from a given tensor other, metrically associated, tensors. Thus $g^{\iota\kappa} B_\kappa = B^\iota$ is a contravariant vector, conjugate to B_ι. The conjugate of

* The clause 'symmetrical' has been inserted, for if $B^{\iota\kappa}$ happened to be a skew tensor, $g_{12} B^{12} = -g_{21} B^{21}$, so that all terms in the complete product would cancel in pairs, giving the uninteresting invariant $g_{\iota\kappa} B^{\iota\kappa} = 0$.

the conjugate can at once be shown to be identical with the original vector. Again, $g^{\iota\kappa} A_{\iota\kappa}$ will be the metrical invariant of the tensor $A_{\iota\kappa}$, and so on.

It will be well to recall here the important property

$$g_{\iota\alpha} g^{\iota\beta} = \delta_\alpha^\beta, \quad . \quad . \quad . \quad . \quad (11)$$

where δ_α^β is the Kronecker symbol (usually written $\delta_{\alpha\beta}$), i.e. 1 or 0 according as $\alpha = \beta$ or $\alpha \neq \beta$. It is itself a tensor, mixed, of rank two, and may as well be denoted by g_α^β. It is the only tensor (of rank two) having in all coordinates the same components 0, 1, apart, of course, from a common scalar factor. The *angle* θ between two comitial infinitesimal vectors dx_ι, dy_ι is defined by the invariant

$$\cos\theta = \frac{g_{\iota\kappa}\, dx_\iota\, dy_\kappa}{du\,.\,dv}, \quad . \quad . \quad . \quad (12)$$

where $du^2 = g_{\iota\kappa} dx_\iota dx_\kappa$, $dv^2 = g_{\iota\kappa} dy_\iota dy_\kappa$ are the norms or squared sizes of these vectors. Thus, orthogonality is expressed by $g_{\iota\kappa} dx_\iota dy_\kappa = 0$. If J is the Jacobian (determinant) $\left| \frac{\partial x_\iota}{\partial x'_\kappa} \right|$ corresponding to the passage from the x to the x' coordinates, the determinant g is easily shown to be transformed into

$$g' = J^2 g, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whence it follows, by a known theorem on transformation of integration variables that the (quadruple) integral

$$\int \sqrt{g}\, dx \text{ (where } dx \text{ stands for } dx_1\, dx_2\, dx_3\, dx_4) \quad (14)$$

extended over any world region is an *invariant* of, or metrically impressed upon, that region. This,

apart from a numerical factor such as $\sqrt{-1}$ (to ensure reality), is called *the volume* of the region or portion of the world.

Again, and this is especially important, with the aid of the metrical tensor, $g_{\iota\kappa}$ or $g^{\iota\kappa}$, an unlimited number of new tensors can be derived from given ones by appropriate differentiations. Here it will be enough to quote some of the most useful differential tensors without stopping to prove their tensorial character. This the reader can supply from any text-book on general relativity or of abstract tensor analysis.

Thus, the most simple (though not the oldest-known), and nowadays perhaps the most fundamental, metrically differential tensor is that discovered (in 1869) by Christoffel, the precursor of modern Tensor Analysis. This is *the covariant derivative* of a covariant vector A_ι, which is again a covariant tensor, of rank two, and can briefly be written

$$A_{\iota\kappa} = \frac{\partial A_\iota}{\partial x_\kappa} - \left\{ {\iota\kappa \atop \lambda} \right\} A_\lambda, \quad . \quad . \quad . \quad (15)$$

where the coefficients of the A_λ are Christoffel symbols of the second kind defined by

$$\left\{ {\iota\kappa \atop \lambda} \right\} = g^{\lambda\mu} \left[{\iota\kappa \atop \mu} \right] = \left\{ {\kappa\iota \atop \lambda} \right\}, \quad . \quad . \quad . \quad (16)$$

the square brackets being Christoffel symbols of the first kind, viz.

$$\left[{\iota\kappa \atop \mu} \right] = \frac{1}{2}\left(\frac{\partial g_{\mu\iota}}{\partial x_\kappa} + \frac{\partial g_{\kappa\mu}}{\partial x_\iota} - \frac{\partial g_{\iota\kappa}}{\partial x_\mu} \right) = \left[{\kappa\iota \atop \mu} \right]. \quad . \quad (17)$$

Notice that both symbols are symmetrical in the two upper indices. Also that *neither* symbol is a tensor.

Similarly,

$$B^{\iota\lambda} = g^{\kappa\lambda}\left[\frac{\partial B^{\iota}}{\partial x_{\kappa}} + \left\{{\alpha\kappa \atop \iota}\right\} B^{\alpha}\right] \quad . \quad . \quad (18)$$

is *the contravariant derivative* of the contravariant vector B^{ι}.

There is an intimate relation between these covariant and contravariant differentiations and the '*parallel shift*' of a vector, a concept and a process discovered by Levi-Civita. This is also extremely interesting and beautiful on its own account. Yet in view of space limitations (and in order not to deviate too much the reader's attention from the main subject) it will not be dwelt upon in this book.

Similar differentiations of tensors of the second and higher orders offer no serious difficulties. It will be enough to mention that if $A_{\iota\kappa}$ be any covariant second-rank tensor,

$$\frac{\partial A_{\iota\kappa}}{\partial x_{\lambda}} - \left\{{\iota\lambda \atop \alpha}\right\} A_{\alpha\kappa} - \left\{{\kappa\lambda \atop \alpha}\right\} A_{\iota\alpha} \quad . \quad . \quad (19)$$

is again a tensor, covariant, of rank three, the covariant derivative of $A_{\iota\kappa}$. It may be well to memorize in this connexion that the covariant derivative of the metrical tensor itself vanishes identically,

$$g_{\iota\kappa\lambda} \equiv 0. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Two more differential tensors, always implying

$g_{\iota\kappa}$ or its determinant, are: *the divergence of a six-vector* (skew tensor) $A^{\iota\kappa}$,

$$A^{\iota} \equiv \mathrm{Div}\,(A^{\iota\kappa}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_{\kappa}}(\sqrt{g}\,A^{\iota\kappa}), \quad . \quad (21)$$

which is a contravariant vector, and *the scalar divergence of a vector B^{κ}*,

$$\mathrm{div}\,(B^{\kappa}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_{\kappa}}(\sqrt{g}\,B^{\kappa}), \quad . \quad . \quad (22)$$

an invariant, of course.

It is useful to introduce for the covariant differentiation a brief symbol. Let this be

$$\mathfrak{D}_{\kappa} = \frac{\partial}{\partial x_{\kappa}} - \frac{\delta}{dx_{\kappa}},$$

where δ is the symbol of variation of a component due to an infinitesimal parallel shift in Levi-Civita's sense of the word. (Cf., for instance, my *Theory of Relativity*, 2nd ed., p. 352, where the awkward 'divisor' dx_{κ} is fully explained.) Such being our convention, $\mathfrak{D}_{\lambda}\,\mathfrak{D}_{\kappa}$ will, of course, stand for an iteration of this operation. And it is important to notice that covariant differentiations are, generally, *non-commutative.*

After this preamble we may explain here in a few lines the structure and the significance of a very important metrically differential tensor, in fact the oldest, historically (viz. of 1861, when it was discovered by Riemann in connexion with investigations on heat conduction). This is the now commonly so-called *Riemann-Christoffel tensor*, which, however,

may more briefly be called *the curvature tensor* of the manifold, in our case of spacetime or, better, of the metrical field $g_{\iota\kappa}$ impressed upon it. It is a mixed fourth-rank tensor, say $R^{\alpha}_{\iota\kappa\lambda}$, and can be most briefly defined by writing down the equation

$$(\mathfrak{D}_\lambda \mathfrak{D}_\kappa - \mathfrak{D}_\kappa \mathfrak{D}_\lambda) A_\iota = R^{\alpha}_{\iota\kappa\lambda} A_\alpha , \quad . \quad (23)$$

which is to hold good for any (arbitrary) covariant vector A_ι. This, when fully developed, in terms of Christoffel symbols, gives for the curvature tensor the expression

$$R^{\kappa}_{\iota\alpha\beta} = \frac{\partial}{\partial x_\alpha}\left\{ {\iota\beta \atop \kappa} \right\} - \frac{\partial}{\partial x_\beta}\left\{ {\iota\alpha \atop \kappa} \right\} + \left\{ {\alpha\gamma \atop \kappa} \right\}\left\{ {\iota\beta \atop \gamma} \right\} - \left\{ {\beta\gamma \atop \kappa} \right\}\left\{ {\iota\alpha \atop \gamma} \right\}. \quad (24)$$

It can be shown that this mixed tensor is antisymmetrical in its lower indices α, β. Bernhard Riemann's own system of so-called *four-index symbols* $(\iota\mu, \lambda\kappa)$ is a purely covariant tensor, metrically associated with the last-written one, viz.

$$(\iota\mu, \lambda\kappa) = R_{\iota\mu\lambda\kappa} = g_{\mu\alpha} R^{\alpha}_{\iota\kappa\lambda} , \quad . \quad . \quad (25)$$

whence follows $R^{\alpha}_{\iota\kappa\lambda} = g^{\mu\alpha}(\iota\mu, \lambda\kappa)$. The latter tensor, denoted by $\{\iota\alpha, \lambda\kappa\}$, was also known, and used in pure geometry, a good many years, viz. by Bianchi, and others. Riemann's symbols, (25), are antisymmetric in κ, λ, that is, $(\iota\mu, \kappa\lambda) = -(\iota\mu, \lambda\kappa)$; three more identical relations hold, $(\iota\mu, \kappa\lambda) = -(\mu\iota, \kappa\lambda)$, $(\iota\mu, \kappa\lambda) = (\kappa\lambda, \iota\mu)$, and $(\iota\mu, \kappa\lambda) + (\iota\lambda, \mu\kappa) + (\iota\kappa, \lambda\mu) = 0$. Such being the case, it can be shown that there are, in an n-fold, $\frac{1}{12} n^2 (n^2 - 1)$, and thus, in our world, only

$$\tfrac{1}{12} 4^2(4^2 - 1) = 20$$

essentially different Riemann symbols left. This, therefore, is also the number of independent components of the curvature tensor $R^{a}_{\iota\kappa\lambda}$. Notice in passing that as urface (twofold) has but *one*, and a three-space *six* such curvature components, and that, in the former case, the essentially unique component, divided by g, say

$$K = \frac{(12,\ 12)}{g}, \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

is *the Gaussian curvature* of the surface, familiar from differential geometry. Needless to say, this 'curvature' (originally defined as the reciprocal product of the two principal curvature radii) is an *intrinsic* property of the twofold itself and does by no means require us to imagine that that surface is 'curved in some three- or more dimensional space'.* In fact, K can be evaluated by purely intrinsic, bi-dimensional observations or measurements, as, e.g., by measuring the excess of the angle sum in a comparatively small geodesic triangle and dividing it by its area. As to the invariance of the curvature of a surface (with respect to any transformations of the coordinates), as given by (26), this follows at once by noticing that (26) is proportional to

$$g^{\iota\mu} g^{\lambda\kappa} (\iota\mu,\ \lambda\kappa) = g^{\iota\mu} g^{\lambda\kappa} R_{\iota\mu\lambda\kappa} = R, \text{ say,}$$

* This trivial remark has seemed worth making here. In fact, I well remember that some five years ago a university professor of theoretical physics asked me, after a lecture on these and allied questions: 'If, as you said, space is curved, *in what is it curved?*' I retorted, of course, that it need not have anything to be curved in.

and this is the invariant of the curvature tensor, for any number of dimensions.

But to return from this bi-dimensional example to our fourfold, the spacetime. In this case, as we saw, the curvature properties can no longer be expressed by a single magnitude but require for their complete description the whole curvature tensor $R^{\kappa}_{\iota\alpha\beta}$ or the associated system of Riemann symbols; in either case twenty independent magnitudes. The familiar concept of Gaussian curvature must now be replaced by that of *Riemannian curvature*. Even this has been introduced, for manifolds of any dimensionality, quite a long time ago. Here, therefore, it will be enough to give a terse explanation of Riemann's concept.

Consider two fixed vectors du_ι, dv_ι and the pencil of vectors

$$dx_\iota = adu_\iota + bdv_\iota,$$

all co-initial at $O(x)$, an arbitrary worldpoint. With each of these as initial (directed) element, draw a geodesic line (defined by $\delta \int ds = 0$; cf. *infra*). Thus a *geodesic surface* will be generated, determined by O itself, one of its points, and by its orientation, which can be taken as given through the oriented surface element

$$\sigma^{\alpha\beta} = du_\alpha dv_\beta - du_\beta dv_\alpha$$

or, equivalently, through the local surface *normal* to be marked by ν. The metrical tensor, say

$$h_{\iota\kappa}\ (h_{11}, h_{12} = h_{21}, h_{22})$$

of this surface as sub-manifold of the world will be

determined by the tensor $g_{\iota\kappa}$ of the latter. Now, with that tensor, $h_{\iota\kappa}$, imagine the Christoffel symbols, of both kinds, and the Riemann symbols constructed, distinguishing them by the subscript h. Then the ordinary, Gaussian, curvature of the geodesic surface at the point O will be, as in (26),

$$K_\nu = \frac{(12,\ 12)_h}{h}, \quad h = |h_{\iota\kappa}| = h_{11} h_{22} - h_{12}^2.$$

Well, it is exactly this perfectly familiar entity which is defined by Riemann as *the curvature of the manifold* (in our case, the world) *for the orientation* ν of the geodesic surface at O, or of its infinitesimal portion (element) $\sigma^{\alpha\beta}$. In other words, Riemannian curvature is the totality of Gaussian curvatures K_ν at the point under consideration. To complete the inquiry it is enough to express $(12,\ 12)_h$ and the determinant h in terms of $g_{\iota\kappa}$ and the pair of vectors du_ι, dv_ι fixing the orientation ν. The result is

$$K_\nu = \frac{(\iota\lambda,\ \kappa\mu)\,\sigma^{\iota\lambda}\,\sigma^{\kappa\mu}}{(g_{\iota\kappa} g_{\lambda\mu} - g_{\iota\mu} g_{\lambda\kappa})\,\sigma^{\iota\kappa}\,\sigma^{\kappa\mu}}, \quad . \quad . \quad (27)$$

to be summed only over $\iota < \lambda$ and $\kappa < \mu$. Whence we see that the vanishing of all symbols $(\iota\lambda,\ \kappa\mu)$, and therefore also of all components of $R^\mu_{\iota\kappa\lambda}$, is *the sufficient condition for homaloidal* (Euclidean) behaviour of the manifold, the curvature K_ν vanishing for all orientations, and all geodesic surfaces being Euclidean planes or developable upon these. This is equivalent to a famous theorem due to Lipschitz. That the vanishing of all Riemann symbols $(\iota\lambda,\ \kappa\mu)$, twenty in

our case, is also the *necessary* condition for a homaloidal manifold, is obvious.

In general, a manifold, e.g. our spacetime, especially in the presence of matter, may be *anisotropic* as well as *non-homogeneous* with respect to its curvature properties, i.e. K_ν a function of position and of ν. If, however, K_ν happens to be isotropic at every point, then the curvature has also, as has been shown by F. Schur, the same numerical value K throughout the manifold. Thus, by (27), the necessary and sufficient condition for *isotropy as well as constancy* of Riemannian curvature becomes

$$(\iota\lambda, \kappa\mu) = K(g_{\iota\kappa}g_{\lambda\mu} - g_{\iota\mu}g_{\kappa\lambda}), \quad . \quad . \quad (28)$$

which amounts to $\frac{1}{12}n^2(n^2-1)$, and in our case twenty, independent equations. In terms of the mixed curvature tensor, by (25), this condition is

$$R^{\alpha}_{\iota\kappa\lambda} = K(\delta^{\alpha}_{\kappa}g_{\iota\lambda} - \delta^{\alpha}_{\lambda}g_{\iota\kappa}), \quad . \quad . \quad (28a)$$

K being a given constant.

To close these lengthy sections on curvature, we may note that the tensor $R^{\alpha}_{\iota\kappa\lambda}$, being mixed, can be contracted, say with respect to α, λ, giving the (symmetrical) second-rank tensor

$$R_{\iota\kappa} = R^{\alpha}_{\iota\kappa\alpha}, \quad . \quad . \quad . \quad . \quad (29)$$

which, together with its invariant $R = g^{\iota\kappa}R_{\iota\kappa}$, identical with the aforesaid $g^{\iota\mu}g^{\lambda\kappa}(\iota\mu, \lambda\kappa)$, is of considerable utility in connexion with (relativistic) world-questions.

Before returning to our main subject (and especially to our next task, that of determining the

properties of the world *on the whole*) we must rapidly recall that all this world-geometry as developed in the preceding sections would be a mere abstract mathematical (logical) science, of considerable beauty, no doubt, but without any interest to the physicist or the astronomer, were it not for the two bridges (built by Einstein) connecting it, as it were, with physics or transforming it into what the logicians call a 'concrete representation' of that lofty, abstract geometry. These bridges, or physical interpretations of the geometrical properties of our manifold, are the two (fundamental) laws or postulates of General Relativity:

(I) $ds = 0$ *to express light propagation*,

and

(II) $\delta \int ds = 0$ *to be the law of motion of a free particle.*

In words: (I) *The minimal* (*or singular*) *lines of space-time are lightlines*, and (II) *The geodesics, in general*,* *are worldlines of free particles.* All further explanations as to the meaning of these postulates and their implications can be read up in any good text-book on general relativity.

We are now ready to proceed with our subject, i.e. to fix the metrical tensor $g_{\iota\kappa}$ for the world- or space-time at large. This will occupy our attention in the next part of the book.

* Since it can be shown that a minimal line is but the limit of a free particle's worldline, viz. when its speed tends to that of light.

PART II

SPACETIME AT LARGE. EINSTEIN'S AND DE SITTER'S SOLUTIONS

WE were trying, in Part I, with the least possible bias, to impress upon the amorphous world a tensor converting it, on the whole, into a metrical spacetime, viz. giving size or norm,

$$ds^2 = g_{\iota\kappa} dx_\iota dx_\kappa, \quad . \quad . \quad . \quad . \qquad (10)$$

to any vector, and herewith also to other higher tensors. The situation, thus far developed, is this: We have, with good reason, made up our mind to accept the *homogeneity of spacetime* on the whole or 'at large', that is to say, in regions remote from massive bodies, and the question as it now stands is: What is the tensor $g_{\iota\kappa}$ which will ensure this behaviour?

Now, if we knew, or had good reasons to assume, that spacetime is not only homogeneous but *isotropic* as well, the answer would—so to speak—jump into the eyes. For then, in fact, would we know, by Schur's theorem, that spacetime is a fourfold of constant curvature. But this we do not know. For, according to that theorem an everywhere isotropic manifold is also homogeneous as to curvature, but a homogeneous one *need not* necessarily be isotropic. In fact, the curvature invariant $R = g^{\iota\kappa} R_{\iota\kappa}$ may well

have the same value throughout the manifold, and yet can the Riemannian curvature K_ν depend, everywhere, on orientation.

Now, of space (three-space) alone, and always at large, we certainly have good reasons to assume that it is not only homogeneous but isotropic as well. But to assert the same thing of that modern 'union' of Space and Time, the world or spacetime, we scarcely have any physical reason at all, except the longing for 'symmetry'—which, however, is a purely mathematical or, if one so prefers, aesthetical requirement. This, however, is not meant to be a book on Aesthetics but one on Cosmology or the actual Universe as a whole.

In fine, we do *not* know that spacetime, even 'on the whole', is isotropic. On the contrary, for the unbiased mind there is scarcely a pair of things more different than the (or a) space-axis and the time-axis,* considered as two 'orientations' ν implied in K_ν. Yet, what primitively (psychologically) differs may, by drill and education, become comparable, nay equivalent, coordinated to each other. And we will therefore leave an open mind as to *the isotropy of spacetime* as a possibility, that is to say, as an assumption whose consequences may perhaps be found not to clash with experiment and observation.

* Just think of (1) passing 'in no time' from here to Andromeda (nebula); (2) waiting for the coming of a friend an aeon, where you are.

As a matter of fact, there are in our days but two serious rival theories abroad, of spacetime at large, that is, and of these one (de Sitter's) *is*, and the other (Einstein's) is *not* isotropic. We will come to know both in some detail presently.

To sum up the remarks thus far advanced, we may say that the problem consists in determining $g_{\iota\kappa}$ so as to make the world *homogeneous, at any rate, and possibly also isotropic.* This, then, will be our requirement to be imposed upon the metrical tensor, and will have to be used for fixing it.

Here, however, my readers, those at least who are conversant with Einstein's general relativity theory, might object, claiming that the determination of the $g_{\iota\kappa}$ at large should be such as to satisfy Einstein's differential *equations of the gravitational field*, i.e. with the (1916) cosmological amplification and with $T_{\iota\kappa}$ written for *the tensor of matter*,*

$$R_{\iota\kappa} - \lambda g_{\iota\kappa} = -\kappa\left(T_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} T\right) \qquad (29)$$

For, in accordance with the spirit of that powerful theory, the 'world on the whole', or a region devoid (approximately) of gravitational pulls, should be treated just as a particular case of the general gravitational field. And if so, our $g_{\iota\kappa}$ would have to be determined so as to satisfy these equations (*ten* in number).

* For details see, e.g., my *Theory of Relativity*, 2nd ed., p. 478. In (29) the coefficient λ of 'the cosmological term', as Einstein calls this 1916-innovation, is meant to be a *constant* (independent of x_1, x_2, x_3, x_4), but so far left undetermined as to its numerical value.

The rejoinder to such an objection, however, is that we are so confident in the world's homogeneity and, to a lesser extent, in its isotropy, that we prefer to build up the desired tensor $g_{\iota\kappa}$ in conformity with this property and then only find out whether and how * the gravitational field equations can be satisfied by this tensor.

Now, if *homogeneity* only is assumed, without yet prejudicing the question of isotropy, we could claim only the constancy (in a certain coordinate system) of the twenty independent components of the full curvature tensor $R^{\alpha}_{\iota\kappa\lambda}$ as given by (24) or, less stringently (say, with the excuse that these are not separately observable), of the ten components of *the contracted* curvature tensor, which can be written (*Theory of Relativity*, pp. 386 et seq.), say, in orthogonal geodesic coordinates,

$$R_{\iota\kappa} = \frac{\partial}{\partial x_\kappa}\left\{{\iota\alpha \atop \alpha}\right\} - \frac{\partial}{\partial x_\alpha}\left\{{\iota\kappa \atop \alpha}\right\}$$
$$= \frac{1}{2}\left[\frac{\partial^2 g_{\alpha\alpha}}{\partial x_\iota \partial x_\kappa} - \frac{\partial}{\partial x_\alpha}\left(\frac{\partial g_{\kappa\alpha}}{\partial x_\iota} + \frac{\partial g_{\iota\alpha}}{\partial x_\kappa}\right) + \frac{\partial^2 g_{\iota\kappa}}{\partial x_\alpha^2}\right], \quad (30)$$

to be summed over α, of course. Thus we should have, for the determination of $g_{\iota\kappa}$,

$$\frac{\partial^2 g_{\alpha\alpha}}{\partial x_\iota \partial x_\kappa} - \frac{\partial}{\partial x_\alpha}\left(\frac{\partial g_{\kappa\alpha}}{\partial x_\iota} + \frac{\partial g_{\iota\alpha}}{\partial x_\kappa}\right) + \frac{\partial^2 g_{\iota\kappa}}{\partial x_\alpha^2} = c_{\iota\kappa},$$

where $c_{\iota\kappa}$ are (ten independent) constants or, subjecting (without any loss to generality) the expres-

* Say, for what value of the parameter λ which, thus far, is at our disposal.

sions $g^{*}_{\kappa a} = g_{\kappa a} - \frac{1}{2}\delta^{a}_{\kappa} g_{\nu\nu}$ to the four conditions $\partial g^{*}_{\iota a}/\partial x_a = 0$,

$$\frac{\partial^2 g_{\iota\kappa}}{\partial x_a^2} = c_{\iota\kappa} \quad \text{(in imaginary coordinates, see loc. cit.)}$$

or, in real coordinates,

$$\left(\nabla^2 - \frac{\partial^2}{\partial x_4^2}\right) g_{\iota\kappa} = c_{\iota\kappa}, \quad . \quad . \quad . \quad (31)$$

where ∇^2 is the usual symbol of the Laplacean $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, so that $\square = \nabla^2 - \partial^2/\partial x_4^2$ is the Dalembertian, familiar from the wave equation. We might reduce even this apparently simple system of (ten) equations, viz. by putting $c_{\iota\kappa} = 0$ for $\iota \neq \kappa$ (again without actual loss to generality), which would leave us with only four, mutually independent, equations

$$\square g_\iota = c_\iota, \quad . \quad . \quad . \quad . \quad (31a)$$

with four constants $c_\iota = c_{\iota\iota}$, $\iota = 1, 2, 3, 4$, for the four tensor components $g_\iota = g_{\iota\iota}$. Each of these is the well-known wave equation (made non-homogeneous by the accession of the constants c_ι), which has been completely integrated a long time ago, so that the desired solution would seem to be ready for use. Unfortunately, however, that system of equations is only *apparently* simple, for the geodesic coordinates x_1, &c., used in this investigation are, ultimately, *local coordinates* (cf. loc. cit., p. 387, and Chapter XI), performing their duty just only at the contemplated worldpoint, their origin, and are, in general, to be

replaced by others and others when one passes to distant regions of the manifold.

This result, based on the assumption of *homogeneity alone*, could not carry us very far in our quest, and the corresponding determination of the $g_{\iota\kappa}$ can be more easily accomplished by solving the gravitational equations (as, historically, has been the case). This will be explained presently.

If, on the other hand, *isotropy* (implying also homogeneity) of the four-dimensional world at large is assumed, the solution sought for follows almost at a glance. In fact, as we saw, every isotropic manifold of Riemannian curvature $K_\nu = K =$ const. is characterized by the differential equations (28) or (28 a) as necessary and sufficient conditions, twenty in number, for the ten unknowns $g_{\iota\kappa}$. If we contract (28 a), writing $R^{a}_{\iota\kappa a} = R_{\iota\kappa}$, the result will, clearly, still represent a necessary but in general not a sufficient condition of isotropy. Keeping this well in mind, contract (28 a) putting $\lambda = a$ and noticing that $\delta^a_a = 1+1+1+1 = 4$, while $\delta^a_\kappa g_{\iota a}$ is just $g_{\iota\kappa}$. Then the result will be, for all combinations of ι, κ,

$$R_{\iota\kappa} = -3Kg_{\iota\kappa}, \quad . \quad . \quad . \quad . \quad (32)$$

which reads: the contracted curvature tensor of an isotropic world is proportional to its metrical tensor. Now, apply to (32) the magic process of contraction once more. Since $g^{\iota\kappa}g_{\iota\kappa} = 4$, this will give, for the curvature invariant ($R = g^{\iota\kappa} R_{\iota\kappa}$),

$$R = -12K; \quad . \quad . \quad . \quad . \quad (33)$$

the curvature invariant is thus seen to be twelve times the negatived Gaussian curvature (K) of any geodesic surface laid through the isotropic world, and *the necessary condition* of its isotropy, implying also homogeneity, becomes

$$R_{\iota\kappa} = \tfrac{1}{4} R g_{\iota\kappa}, \quad R = \text{const.}, \quad . \quad . \quad (34)$$

or, yet more simply,

$$R_\iota^\kappa = \tfrac{1}{4} \delta_\iota^\kappa R,$$

which reads, explicitly,

$$R_1^1 = R_2^2 = R_3^3 = R_4^4 = \tfrac{1}{4} R, \quad . \quad . \quad (34\,a)$$

all other components of this mixed tensor

$$(R_\iota^\kappa = g^{\kappa a} R_{\iota a})$$

being nil. The more stringent, *sufficient* condition of isotropy, (28 a), can now be written

$$R^a_{\iota\kappa\lambda} = \frac{R}{12} (\delta^a_\lambda g_{\iota\kappa} - \delta^a_\kappa g_{\iota\lambda}). \quad . \quad . \quad (35)$$

Taking first the necessary condition, introduce the values (34) of $R_{\iota\kappa}$ into the general equations (29) of the gravitational field without yet prejudicing the question whether the tensor of matter $T_{\iota\kappa} = T^0_{\iota\kappa}$ does or does not vanish at large, or outside of lumps of condensed matter. Then

$$(\tfrac{1}{4} R - \lambda - \frac{\kappa}{2} T^0)\, g_{\iota\kappa} = -\kappa T^0_{\iota\kappa},\text{*}$$

whence, multiplying both sides by $g^{\iota\kappa}$, contracting, and recalling that $g^{\iota\kappa} g_{\iota\kappa} = 4$,

$$R - 4\lambda = \kappa T^0,$$

* The factor κ is the gravitation constant multiplied by 8π and divided by the square of the 'light velocity', 3.10^{10} cm./sec., say $\kappa = 8\pi k/c^2$.

and if this be introduced into the last equation,

$$g_{\iota\kappa} = 4T^0_{\iota\kappa}/T^0.$$

Notice that, R and λ being constant, so also must, at any rate, be T^0, the scalar of the material tensor which (apart from niceties) is simply proportional to the density of matter, or of energy. This constancy of T^0 might, of course, be predicted from the assumed homogeneity of the world at large.

To sum up the last results, we have, as *the necessary* conditions of isotropy,

$$g_{\iota\kappa} = \frac{4R_{\iota\kappa}}{R} = \frac{4T^0_{\iota\kappa}}{T^0} \quad . \quad . \quad . \quad (36)$$

and

$$\lambda = \tfrac{1}{4}(R - \kappa\rho_0), \quad . \quad . \quad . \quad . \quad (37)$$

where ρ_0, the density at large, has been written for T^0.

Thus, if the tensor of matter, and therefore also its scalar ρ_0, vanish at large, Einstein's gravitational equations are fully satisfied by taking

$$\lambda = \tfrac{1}{4}R, \quad . \quad . \quad . \quad . \quad (38\,s)$$

while the last term in (36) is and remains indeterminate (0/0), and we are left for the determination of the $g_{\iota\kappa}$ with the equations

$$\frac{R_{\iota\kappa}}{R} = \tfrac{1}{4}g_{\iota\kappa}.$$

These are ten differential equations, of the second order, for as many $g_{\iota\kappa}$. Without any loss to generality we can choose the coordinates so as to annihilate all but the diagonal g's, which can be written briefly

$g_{11} = g_1$, &c., $g_{44} = g_4$. Such also will then be the case of the $R_{\iota\kappa}$, and the system will be reduced to the four equations

$$R_\iota = \frac{R}{4} g_\iota \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

To satisfy these, it will be found sufficiently general to put (in quasi-polar coordinates x, ϕ, θ, and ct as x_1, x_2, x_3, x_4)

$$g_1 = g_1(x),\ g_2 = -x^2,\ g_3 = -x^2 \sin^2\phi,\ g_4 = g_4(x), \quad (40)$$

where g_1, g_4 are functions of x alone, whose form is to be determined. This is equivalent to writing, for the line-element,

$$ds^2 = g_1 dx^2 - x^2(d\phi^2 + \sin^2\phi\, d\theta^2) + g_4 c^2 dt^2. \quad (40')$$

With these values, and the abbreviations $h_1 = \log g_1$, $h_4 = \log g_4$, the components of the contracted curvature tensor become

$$R_1 = \frac{h_4''}{2} + \frac{h_4'}{4}(h_4' - h_1') - \frac{h_1'}{x},$$

$$R_2 = \frac{R_3}{\sin^2\phi} = -1 - \frac{1}{g_1}\left\{1 + \frac{x}{2}(h_4' - h_1')\right\},$$

$$R_4 = \frac{g_4}{g_1}\left\{R_1 + \frac{h_1' + h_4'}{x}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where $h_1' = dh_1/dx$, &c. The equations (39), with

$$R = -12K = +\frac{12}{\mathfrak{R}^2},^* \quad . \quad . \quad . \quad (42)$$

now assume the form

$$(a) \quad \frac{h_4''}{2} + \frac{h_4'}{4}(h_4' - h_1') - \frac{h_1'}{x} = \frac{3}{\mathfrak{R}^2} g_1,$$

* This by (33), and with $\mathfrak{R}$ written for *the curvature radius.*

$$(b)\quad 1+\frac{1}{g_1}\left\{1+\frac{x}{2}(h_4'-h_1')\right\}=\frac{3x^2}{\Re^2}$$

$$\left(R_3=\frac{R}{4}g_3 \text{ says the same thing}\right),$$

$$(c)\quad R_1+\frac{1}{x}(h_1'+h_4')=\frac{3}{\Re^2}g_1.$$

From (*a*), (*c*), follows at once $h_1'+h_4'=0$ and $g_1g_4=$ const., and since a constant factor can always be thrown upon the time unit, one may as well write

$$g_1g_4=-1.$$

Thus (*b*) becomes

$$\frac{d}{dx}(g_4-1)+\frac{1}{x}(g_4-1)+\frac{3x}{\Re^2}=0, \quad . \quad (43)$$

which is of the well-known form of Euler's equation and has for its most general or complete solution

$$g_4=1-\frac{x^2}{\Re^2}-\frac{2L}{x}, \quad . \quad . \quad . \quad (44)$$

where $2L$ is an arbitrary constant. This general integral is of interest in connexion with a mass-centre, giving in fact the gravitational field around such a centre or say a spherical body, outside the body itself. It has, accordingly, a singularity at $x=0$; the origin of the space coordinates. It may be useful in the sequel. At the present moment, however, we require the (metrical, or gravitational) *field at large*, and are thus concerned only with the particular solution for $L=0$.

This is

$$g_4=1-\frac{x^2}{\Re^2},$$

and gives also at once the remaining unknown $g_1 = -1/g_4$. (That all *three* equations, (a), (b), (c), are satisfied by these g_1, g_4, can readily be verified.) It is convenient to put

$$x = \mathfrak{R} \sin \sigma, \quad \sigma = \frac{r}{\mathfrak{R}},$$

both, $\mathfrak{R}$ and r, having the dimensions of a length, the latter of which will play the role of the radial distance from the origin. With this new variable we have

$$g_4 = \cos^2 \sigma, \quad g_1 = -\frac{1}{\cos^2 \sigma}.$$

Ultimately, therefore, the required metrical tensor is, by (40),

$$g_1 = -\frac{1}{\cos^2 \sigma}, \quad g_2 = -\mathfrak{R}^2 \sin^2 \sigma, \ g_3 = g_2 \sin^2 \phi,$$
$$g_4 = \cos^2 \sigma, \qquad (45')$$

or, equivalently, the line-element, since $g_1 \, dx^2 = -dr^2$,

$$ds^2 = \cos^2 \sigma \, . \, c^2 \, dt^2 - \{dr^2 + R^2 \sin^2 \sigma \, (d\phi^2 + \sin^2 \phi \, d\theta^2)\},$$
$$\sigma = \frac{r}{\mathfrak{R}}, \qquad (45)$$

or

$$ds^2 = \cos^2 \sigma \, . \, c^2 \, dt^2 - dl^2,$$

where $dl^2 = dr^2 + \mathfrak{R}^2 \sin^2 \dfrac{r}{\mathfrak{R}} (d\phi^2 + \sin^2 \phi \, d\theta^2)$ is the well-known line-element of a three-space of constant positive curvature $\mathfrak{R}^{-2}$ or, in technical language, of an *elliptic space* of curvature radius $\mathfrak{R}$.

This metrical tensor, or line-element (45), is de

Sitter's solution, obtained by him,* as an integral of Einstein's gravitational equations, by another method.

The reader will remember that (45) was so constructed as to satisfy only *the necessary* condition for world isotropy, viz. the contracted equations (34). But it so happens that it satisfies also *the sufficient* condition, that is to say, the original set of equations (28 a), re-written in (35). In fact, taking again an orthogonal system of coordinates in which, that is, only $g_1 = g_{11}$, &c., survive, those equations split into four groups,

$$R^a_{\kappa\kappa\kappa} = \frac{R}{12}(\delta^a_\kappa g_\kappa - \delta^a_\kappa g_\kappa) = 0 \quad (\text{for } \iota = \kappa = \lambda),$$

$$R^a_{\kappa\kappa\lambda} = \frac{R}{12}\delta^a_\lambda g_\kappa \quad (\text{for } \iota = \kappa \neq \lambda),$$

$$R^a_{\iota\kappa\kappa} = \frac{R}{12}(\delta^a_\kappa g_{\iota\kappa} - \delta^a_\kappa g_{\iota\kappa}) = 0 \quad (\text{for } \iota \neq \kappa = \lambda),$$

$$R^a_{\iota\kappa\lambda} = 0 \quad (\text{for } \iota, \kappa, \lambda \text{ all different}).$$

Thus the only surviving components of the curvature tensor belonging to an isotropic world are

$$R^\lambda_{\kappa\kappa\lambda} = \frac{R}{12} g_\kappa, \quad \kappa \neq \lambda,$$

not to be summed over the equal indices. Since the right-hand member does not contain the λ, these equations mean simply that

$$R^1_{\kappa\kappa 1} = R^2_{\kappa\kappa 2} = R^3_{\kappa\kappa 3} = R^4_{\kappa\kappa 4} = \frac{R}{12} g_\kappa$$

* W. de Sitter, *Monthly Notices Roy. Astr. Soc.* for Nov. 1917.

and, since $\kappa \neq \lambda$, the values of κ in the first term are 2, 3, 4 only, in the second 3, 4, 1, and so on. Ultimately, therefore, the necessary and sufficient conditions of isotropy are

$$R^1_{221},\ R^1_{331},\ R^1_{441} = (g_2,\ g_3,\ g_4)\,\frac{1}{\Re^2}$$

$$R^2_{332},\ R^2_{442},\ R^2_{112} = (g_3,\ g_4,\ g_1)\,\frac{1}{\Re^2}$$

$$R^3_{443},\ R^3_{113},\ R^3_{223} = (g_4,\ g_1,\ g_2)\,\frac{1}{\Re^2}$$

$$R^4_{114},\ R^4_{224},\ R^4_{334} = (g_1,\ g_2,\ g_3)\,\frac{1}{\Re^2}.$$

It will be enough to show explicitly that two or three of these twelve equations are satisfied by (45′), let us say the first and the last. Now, substituting the tensor (45′), with x, ϕ, θ, ct as x_1, x_2, x_3, x_4, one finds

$$R^1_{221} = -\frac{d}{dx}\left(\frac{x}{g_1}\right) + \left\{{22 \atop 1}\right\}\left[\left\{{21 \atop 2}\right\} - \left\{{11 \atop 1}\right\}\right] = \frac{x g_1'}{2g_1^2}$$

$$= \tfrac{1}{2}\,x g_4' = -\frac{x^2}{\Re^2} = \frac{g_2}{\Re^2},$$

so that the first equation is satisfied. Again,

$$R^4_{334} = -\left\{{41 \atop 4}\right\}\left\{{33 \atop 1}\right\} = \tfrac{1}{2}\,g_4' x \sin^2\phi = -\frac{x^2 \sin^2\phi}{\Re^2} = \frac{g_3}{\Re^2},$$

which is the last equation. Finally, and this has seemed particularly instructive,

$$R^4_{114} = \frac{\partial}{\partial x}\left\{{14 \atop 4}\right\} + \left\{{14 \atop 4}\right\}^2 - \left\{{14 \atop 4}\right\}\left\{{11 \atop 1}\right\}$$

$$= \tfrac{1}{2}\left[h_4'' + \tfrac{1}{2}\,h_4'\,(h_4' - h_1')\right],$$

and since $g_1 g_4 = -1$, and therefore, $h_1' = -h_4'$,

$h_4'' + h_4'^2 = g_4''/g_4$, the tenth of our array of equations becomes

$$g_4'' = -2/\mathfrak{R}^2;$$

its complete solution is $g_4 = A + Bx - x^2/\mathfrak{R}^2$, and since $g_4 = 1$ for $x = 0$,

$$g_4 = 1 - x^2/\mathfrak{R}^2 + Bx,$$

where the integration constant B remains arbitrary. If we put $B = 0$, then $g_4 = 1 - x^2/\mathfrak{R}^2$, as above.

The verification of the remaining ten equations may be left to the reader, though it is scarcely necessary. In fact, a much simpler proof of the complete isotropy of the tensor under discussion can be given by showing that the corresponding line-element, (45), is identical with

$$-ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2, \quad . \quad (45'')$$

where the fifth coordinate is only an auxiliary,* bound to the remaining four by the relation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \mathfrak{R}^2,$$

which is simply the equation of a (hyper) sphere in a Euclidean or homaloidal five-dimensional space, ds being the line-element of this spherical fourfold (Kugelraum, as called by W. Killing). And that such a fourfold is perfectly isotropic is seen without

* The reader will do well to drop it from his mind the moment it has done its analytic service. This fifth coordinate is, to use a Slavic proverb, as superfluous as 'the fifth wheel of a car'. Five-dimensional aspirations at physical or cosmological successes, in which some modern writers, as Th. Kaluza, nay even Einstein, indulge, strike one as naïve. As a matter of fact, these modern excursions into the fifth dimension never bore any substantial fruit.

formal proof. (Details concerning the passage from the form (45) to (45″), or vice versa, will be found on p. 504 of my Relativity book.)

In fine, then, de Sitter's world at large, characterized by the metrical tensor

$$g_1 = -\sec^2\sigma, \quad g_2 = g_3/\sin^2\phi = -\mathfrak{R}^2\sin^2\sigma, \quad g_4 = \cos^2\sigma, \qquad (45')$$

or the line-element

$$ds^2 = \cos^2\sigma \,.\, c^2\,dt^2 - dl^2, \quad . \quad . \qquad (45)$$

impressed upon the primitive spacetime, is *perfectly isotropic,* as well as homogeneous.

The rival solution, Einstein's world, which we will now go on to consider, does not possess this (mathematically) precious property. It is no more isotropic, as a fourfold, than is a cylinder, as a twofold. In fact, most writers refer to it as *the cylindrical world* (while de Sitter's may conveniently be termed a *spherical world,* in spite of the fact that in (45″) three terms only are positive, and the remaining two negative).

Einstein himself has arrived at his world, or its line-element, in the following way.

Having originally claimed for the world at large the special-relativistic behaviour or the Euclidean line-element

$$ds^2 = c^2dt^2 + (dx_1^2 + dx_2^2 + dx_3^3), \qquad x_1, x_2, x_3 = i\,(x, y, z),$$

i.e. the metrical tensor $g_{\iota\kappa} = \delta_\iota^\kappa$, he has about 1916 *

* Guided, in part, by Ernst Mach's principle of Relativity of Inertia.

worked himself up into a radical dislike of this infinite, homaloidal manifold, and not being able to dictate to us satisfactory conditions 'at infinity', cut the matter short by declaring space to be finite, Riemannian, or spherical. Thus did he, at first, simply replace the Euclidean (space-) line-element

$$dr^2+r^2(d\phi^2+\sin^2\phi\,d\theta^2)=dx^2+dy^2+dz^2$$
$$=-(dx_1^2+\dots)$$

by the spherical, more appropriately, elliptic line-element

$$dl^2=dr^2+\Re^2\sin^2\sigma\,(d\phi^2+\sin^2\phi\,d\theta^2),\quad \sigma=r/\Re,$$

with $\Re$ as the curvature radius of that three-space, and wrote down accordingly, in his *Kosmologische Betrachtungen* of 1916,

$$ds^2=c^2dt^2-dl^2, \quad . \quad . \quad . \quad . \quad (46)$$

to hold for the world at large, that is. The procedure was a bare subtraction of an elliptic space-element (squared) from the squared time-element. Having constructed this ds^2 or the equivalent tensor $g_{\iota\kappa}$, he then hastened to support it by, and to harmonize it with, his original gravitational field equations,

$$R_{\iota\kappa}=-\kappa\,(T_{\iota\kappa}-\tfrac{1}{2}g_{\iota\kappa}T).$$

And since (46) would not fit these, he amplified them by inserting 'the cosmological term' $-\lambda g_{\iota\kappa}$, with λ a 'universal constant', thus arriving at the equations quoted above, under (29). This change, necessary to suit a finite, closed space, was actually suggested to him by the modified Newtonian potential $\Omega=e^{-r\sqrt{\lambda}}/r$

used by Carl Neumann and discussed (1896) by Seeliger in connexion with the olden difficulties of an infinitely extended distribution of matter, the correspondingly modified Laplace-Poisson equation being

$$\nabla^2 \Omega - \lambda \Omega = -4\pi k \rho.$$

But let us proceed with the subject. Einstein's amplified equations (29) give at once $R - 4\lambda = \kappa T$, and they can therefore be written

$$R_{\iota\kappa} - \tfrac{1}{2}(R - 2\lambda)\, g_{\iota\kappa} = -\kappa T_{\iota\kappa} . . \quad . \quad (29')$$

Now, Einstein assumed that the tensor $T_{\iota\kappa}$ at large, i.e. outside of condensed matter, is reducible in a certain system, viz. that in which (46) holds, to

$$T_{44} = \rho_0 = \text{const.} . \quad . \quad . \quad . \quad (47)$$

as the only surviving component. But substituting in (29′) this tensor for $T_{\iota\kappa}$, and the values of $R_{\iota\kappa}$ and R corresponding to (46), i.e.

$$R_{ii} = \frac{2}{\Re^2} g_{ii}, \quad R_{44} = 0, \quad R = \frac{6}{\Re^2} \quad (i = 1, 2, 3).$$

one finds the four equations

$$(\lambda - \Re^{-2})\, g_{ii} = 0, \qquad (3/\Re^2 - \lambda)\, g_{44} = \kappa \rho_0, . \quad (48)$$

where the g's are as in (46). Now, the first three of these equations are all satisfied by $\lambda = 1/\Re^2$, and since $g_{44} = 1$, the last equation leads to $2/\Re^2 = \kappa\rho_0$. Thus, Einstein's new equations (29′) assume the form

$$R_{\iota\kappa} - \tfrac{1}{2}\left(R - \frac{2}{\Re^2}\right) g_{\iota\kappa} = -\kappa T_{\iota\kappa},$$

and the curvature of the world at large (which, apart from the sign, is also that of the elliptic space) becomes proportional to the uniform density of matter prevailing on the whole (which Einstein considers as the actual mean density substituted in a kind of approximation for the granular mass distribution, the 'grains' being stars or even nebulae or galaxies), to wit

$$\frac{1}{\Re^2} = \tfrac{1}{2}\kappa\rho_0 = \frac{4\pi k}{c^2}\rho_0 . \quad . \quad . \quad . \quad (49)$$

Notice that the absence of matter at large or, from Einstein's approximate macroscopic point of view, a vanishing mean density, $\rho_0 = 0$, would imply $\Re = \infty$ or an infinite, Euclidean space. Einstein insists, however, on $\rho_0 > 0$.* Since the volume of the elliptic space (of the *polar*, not the antipodal kind as imagined by Einstein) is $\pi^2\Re^3$, *the total mass* of the universe will be, in astronomical units, $M = k\rho_0\pi^2\Re^3$, and therefore, by formula (49),

$$M = \frac{\pi c^2}{4}\Re, \quad . \quad . \quad . \quad . \quad (50)$$

which reads: mass of universe proportional to its radius and, of course, vice versa—a quaint relation which Eddington delights to express by exclaiming: 'The more matter the more room!' This slogan seems to have been picked up by the U.S.A. street-

* He scarcely has any convincing reason for this tendency. In his Princeton Lectures, for instance, he says of $\rho_0 = 0$, that 'although such an assumption is logically possible, *it is less probable*' than that of a finite ρ_0.

car conductors (at least those of Rochester, N.Y.), who please their directors by adopting the principle: The more passengers, the more room—in which they are stoutly supported by the elasticity of American ribs and the general docility of the public.

Apart from jokes, there is of course nothing absurd in this striking relation (50). Another question is, of course, whether it actually holds or whether it can at all be proved or disproved. We will soon find good reasons for believing that the answer is in the negative. Meanwhile let us notice that the elimination of the curvature radius between (49) and (50) yields the equally interesting relation

$$M = \frac{c^2}{8} (\pi/k\rho_0)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (50')$$

The smaller the mean density, the greater the required total mass of the universe, so that an evanescent ρ_0 would, on this scheme, call for $M = \infty$, making at the same time the world Euclidean

$$(ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2)$$

or, as it is sometimes called, homaloidal (which means 'even', not curved). It goes without saying that until a lower limit to ρ_0 is astronomically established, Einstein's 1916-world is as good as indistinguishable from the older homaloidal one.

But let us turn to some other implications of Einstein's cosmology.

First, still keeping away from condensed matter, let us inquire into the laws of motion of free particles

in his world. These laws, as we already know, are expressed by the world-geodesics, i.e. by $\delta\int ds = 0$, where ds is as in (46). Without loss to generality we may confine ourselves to the plane $\phi = \pi/2$, so that

$$ds^2 = c^2 dt^2 - (dr^2 + \Re^2 \sin^2\frac{r}{\Re}\, d\theta^2),$$

whence, using the Lagrangian development of $\delta\int ds$, one finds without trouble, as the required equations of motion,

$$\ddot{r} = \frac{\Re\ddot{\theta}}{2}\sin\frac{2r}{\Re}, \qquad \dot{\theta}\sin^2\frac{r}{\Re} = \text{const.}, \qquad \dot{t} = \text{const.},$$

where the dot denotes d/ds. Since one of these equations is a consequence of the two others, we may use the last two only, substituting $\dot{x}_4 = \text{const.}$ into the line-element. Thus the equations of free motion become

$$\frac{dl}{dt} = v_0, \qquad \Re^2 \sin^2\frac{r}{\Re}\cdot\frac{d\theta}{dt} = h, \quad . \quad . \quad (51)$$

where v_0, h are integration constants, and dl, as before, the line-element of elliptic three-space. Thus, a free particle moves in this space with constant velocity, and its radius vector r sweeps equal areas in equal times t. Eliminating dt, the orbit of the particle is seen to be determined by the differential equation

$$\sin^2\frac{r}{\Re}\cdot\frac{d\theta}{dl} = C,$$

where $C = h/v_0\Re^2 = \text{const.}$ This can readily be

shown to represent a three-geodesic, that is to say, a straight line of the elliptic space.

In fine, a free particle moves in this space along a straight line with uniform velocity.

This implies, of course, that if it is placed anywhere at rest, in Einstein's fundamental frame, and left to itself, it will stay there for ever. The mass distribution being uniform throughout the universe, this result might have been predicted without taking the trouble of writing down the equations of motion.

Again, since a minimal line, which represents light propagation, is but a limiting case of a world-geodesic, (51), light rays, too, are straight lines of the elliptic space and the velocity of light along them is constant, $dl/dt = c$, which follows also at once from $ds^2 = c^2\, dt^2 - dl^2 = 0$.

In this respect, then, Einstein's cylindrical world at large is seen to resemble completely the older, homaloidal one.

A rather curious reflection here suggests itself. Einstein in his 'cosmological contemplations' of 1916 set out with the resolute plan of satisfying what he called 'Mach's principle' (relativity of inertia). Now, from what has just been expounded it will be seen that the validity of the familiar law of inertia in comparatively desolate regions of space is secured by the very structure of Einstein's new world. But is the principle of 'relativity of inertia'* also satis-

* This principle or postulate (pushed by Einstein to the extreme)

fied? Well, it certainly is, though in a rather unexpected way. For, whereas one does not see whether and how the inertia (mass) of a particle is produced by all the remaining matter in the universe, and particularly whether there is enough matter for that purpose, yet the particle owes the very 'room', in which only it can display its inert character, to all that matter.

But let us proceed with the subject. The light rays being rectilinear in the elliptic space, *the parallax* formula will, in Einstein's cosmology, be exactly as that known many years (at least through its hyperbolic counterpart) from works on non-Euclidean geometry, apart—of course—from slight modifications due to relativistic refinement. To wit, if r be the distance of a star, its parallax p will be given by

$$\tan p = \sin\frac{a}{\mathfrak{R}}\cot\frac{r}{\mathfrak{R}}, \quad . \quad . \quad . \quad (52)$$

where a is the mean distance of the earth from the sun or the usual 'astronomical unit' of length (149 million km.). Now, since the curvature radius $\mathfrak{R}$, if at all finite, contains at any rate an enormous number of astronomical units, (52) can as well be written

$$\tan p = \frac{a}{\mathfrak{R}}\cot\frac{r}{\mathfrak{R}}. \quad . \quad . \quad . \quad (52')$$

claims that the inertia of a particle should not only be affected or 'influenced' but also 'conditioned' by, or entirely due to, all the remaining matter existing in the universe, in fine: a joint gravitational effect of all that matter. See Einstein's 'Kosmologische Betrachtungen, &c.', Berlin Acad., *Berichte*, 1917, p. 142.

The classical parallax formula was

$$\tan p = \frac{a}{r}, \quad . \quad . \quad . \quad . \quad (52c)$$

setting, of course, no lower limit to p other than 0, which is approached with increasing distance. Now, such also is the case with (52′), which gives

$$p = 0, \text{ for } r = \tfrac{1}{2}\pi\mathfrak{R},$$

the greatest possible distance apart of two points in elliptic space.

We thus see that there is no possibility of deriving a crucial test, for or against Einstein's cosmology, from these quarters, that is in regions distant enough from lumps of condensed matter.[1]

Such being the case, let us turn, then, to the comparative neighbourhood of some huge piece of condensed matter, as e.g. the sun—to be considered here as a gravitational mass-centre, seeing that the main interest lies in the external field. Let this be taken for the origin of polar coordinates. The field equations outside of condensed matter follow from (29′) by putting $T_{44} = \rho_0 g_{44}$, $T = \rho_0$, and equating all other $T_{\iota\kappa}$ to zero. This, with (49), gives

$$R_{i\kappa} = \frac{2}{\mathfrak{R}^2} g_{i\kappa}, \quad R_{44} = 0, \quad i = 1, 2, 3, \quad . \quad (53)$$

as before, for the particular case of

$$ds^2 = dx_4^2 - dl^2,$$

only that then g_{44} was equal to 1, and $T_{44} = \rho_0$.

[1] For more details cf. my *Relativity*, p. 485.

As we saw before, these equations are satisfied by the tensor corresponding to the line-element

$$ds^2 = dx_4^2 - dl^2$$

itself. This is their simplest solution, symmetrical around *any* point taken as $r = 0$, and thus perfectly homogeneous throughout the manifold. What is now required is a solution of (53) symmetrical around a unique point O only, the seat of the mass-centre under consideration. A sufficiently general form of ds^2 for this purpose is that already used in connexion with de Sitter's world, viz.

$$ds^2 = g_1 dx^2 - x^2 (d\phi^2 + \sin^2 \phi \, d\theta^2) + g_4 c^2 dt^2,$$

where g_1, g_4 are functions of x alone. The contracted curvature tensor, R_{11}, &c., is again as before. Thus, a substitution into (53) gives the three differential equations

$$R_{11} = \frac{h_4''}{2} + \frac{h_4'}{4}(h_4' - h_1') - \frac{h_1'}{x} = \frac{2g_1}{\Re^2} \qquad (\alpha)$$

$$-R_{22} = \frac{1}{g_1}\{1 + \frac{x}{2}(h_4' - h_1')\} + 1 = \frac{2x^2}{\Re^2} \qquad (\beta)$$

$$\frac{g_1}{g_4} R_{44} = R_{11} + \frac{1}{x}(h_1' + h_4') = 0 \qquad (\gamma)$$

for the two unknown functions g_1, g_4 of x. Now, substituting R_{11} from (α) into (γ) and eliminating h_4' between (β) and (γ), we have, as before,

$$f' + \frac{1}{x} f - \frac{3x}{\Re^2} = 0, \quad f = 1 + 1/g_1,$$

whence the solution (having a singularity at $x = 0$, as it ought to)

$$\frac{1}{g_1} = 1 - \frac{2L}{x} - \frac{x^2}{\Re^2}, \quad . \quad . \quad . \quad (54)$$

where L is an integration constant characterizing the masspoint at O (viz. L = its gravitation radius = mass$/c^2$). Equation (γ) now becomes

$$h_1' + h_4' = -2xg_1/\Re^2, \quad \text{so that}$$

$$h_4' = -\frac{2Lg_1}{x^2}, \quad . \quad . \quad . \quad . \quad (55)$$

and the problem would thus be reduced to a mere quadrature, yielding the most general stationary, radially symmetrical field. Unfortunately (for the cylindrical world), however, and unlike the situation in the case of a homaloidal world, the first field-equation, (α), is *not* satisfied automatically by (54), (55). Nay, it clashes with them. For, with these values of g_1 and h_4', equation (α) becomes

$$\frac{Lg_1^2}{\Re^2 x} = 0,$$

and thus could only be satisfied if either $L = 0$ or $\Re = \infty$, the former alternative knocking out the mass-centre, and the latter expanding the cylindrical world into an infinite 'dreary homaloid' (to borrow Clifford's expression, used by him on a different occasion).

In fine, if one is content to accept Einstein's own set of cosmological assumptions, i.e.—

the cosmological term $-\lambda g_{\iota\kappa}$ on the left of his field-equations,

some *non-vanishing* density ρ_0 (as macroscopic mean),

and *no* pressure outside of condensed matter,

and hence also a total mass of the universe

$$M = \text{const.}/\sqrt{\rho_0},$$

one cannot have mass-centres or, for that matter, any massive globes with radially symmetrical fields surrounding them.

But, of course, one could not make a single step without them, especially in celestial mechanics. Is there no way out of this embarrassment? Some ten years ago de Sitter (*Monthly Notices, Roy. Astr. Soc.*, vol. 78, pp. 19–23) believed he had discovered one. He proposed to modify somewhat Einstein's assumptions by admitting an isotropic *pressure*, which—outside of condensed matter—is to keep company with the formerly lonely tensor component $g_{44}\rho_0$, and imagined, in fact, the uncondensed matter to behave as an *incompressible fluid*,* for which there is no other reason than that of mathematical simplicity. Thus, let us write after de Sitter, for the material tensor around our mass-centre,

$$T_{ii} = -g_i p, \quad T_{44} = g_4\rho_0, \text{ and therefore } T = \rho_0 - 3p,$$

where p stands for pressure divided by c^2. Let us see what this gives. But unlike de Sitter, who neglects second-order terms, let us develop the rigorous solu-

* Whereas Einstein's world-matter behaved like dust or a cloud of particles ($p = 0$), at least in his original (1916) paper.

tion, since this will disclose far away at the polar plane $r = \frac{1}{2}\pi\mathfrak{R}$ of the mass-centre, certain very disturbing peculiarities, which in his approximate treatment were simply swept away with the neglected terms.

The equations (α), (β), (γ), with the same general form of ds^2, are now replaced by a set which is easily reducible to

$$(\alpha') \quad (p+\rho_0)\sqrt{g_4} = A,$$

$$(\beta') \quad f' + \frac{1}{x}f - (\kappa\rho_0 + \lambda)\,x = 0,$$

$$(\gamma') \quad \frac{d}{dx}(g_1 g_4) = -\kappa A x g_1^2 \sqrt{g_4},$$

where $f = 1 + g_1^{-1}$, and A is an arbitrary constant. The most general integral of (α') is, with $2L$ written for the integration constant,

$$f = \frac{2L}{x} + \frac{x^2}{\mathfrak{R}_1^2}, \quad \mathfrak{R}_1^2 = \frac{3}{\kappa\rho_0 + \lambda}.$$

If ρ_0, as well as λ, is the same as in the absence of a mass-centre, then, whatever the residual p_0 (i.e. the value of p when $L = 0$),

$$\mathfrak{R}_1^{-2} = \frac{\kappa}{3}(\rho_0\mathfrak{R}^2 + \tfrac{1}{2}\rho_0 + \tfrac{3}{2}p_0) = \frac{\kappa}{2}(\rho_0 + p_0) = \mathfrak{R}^{-2},$$

so that the constant $\mathfrak{R}_1$ is identical with the world-radius $\mathfrak{R}$, and we find again, as in the universe devoid of pressure,

$$-\frac{1}{g_1} = 1 - \frac{2L}{x} - \frac{x^2}{\mathfrak{R}^2}. \quad . \quad . \quad . \quad (56)$$

The general solution of (γ') now becomes, with $A = p_0 + \rho_0 = 2\kappa/\Re^2$,

$$\sqrt{g_1 g_4} = -\frac{1}{\Re^2}\int x g_1^{\frac{3}{2}}\, dx + C, \quad . \quad . \quad (57)$$

where C is, thus far, an arbitrary constant.

The solution of the problem is now contained in (56) and (57), always with the form

$$ds^2 = g_1 dx^2 - x^2 (d\phi^2 + \sin^2\phi\, d\theta^2) + g_4 c^2 dt^2$$

of the line-element. It remains to discuss it, not shrinking from very distant regions of space.

Introduce, as before, the new coordinate $\sigma = r/\Re$ through $x = \Re \sin \sigma$, so that the new g_1 will be (56) times $\cos^2 \sigma$, while g_4 will remain unchanged. Thus, and putting $C = 0$ (cf. *Theory of Relativity*, p. 492), the metrical, and at the same time gravitational, field around the mass-centre at $r = 0$ will become

$$g_1 = -\left(1 - \frac{2L}{\Re}\sec^2\sigma\cos\sigma\right)^{-1}, \; g_2 = \frac{g_3}{\sin^2\phi} = -\Re^2\sin^2\sigma,$$

$$\sqrt{-g_1 g_4} = \cos\sigma \int (-g_1)^{\frac{3}{2}} \frac{\sin\sigma}{\cos^2\sigma}\, d\sigma\,. \quad (58)$$

Now, for small $\frac{r}{\Re}$ or, as one may say, in the planetary neighbourhood of the mass-centre (considered e.g. as a star) this solution offers nothing of interest, its astronomical implications not being distinguishable from those based on Einstein's original homaloidal

world. In fine, this neighbourhood speaks neither for nor against the cylindrical world. But when we turn to the other extreme, to the neighbourhood of the 'horizon'* or, in older geometrical language, *the polar (plane) of the mass-centre*, $\sigma = r/\Re = \frac{1}{2}\pi$, we are faced by a behaviour of that hypothetical world so strikingly inadmissible as to make Einstein's cosmology utterly untenable.

In fact, let us consider σ-values but little smaller than a right angle, say

$$\sigma = \tfrac{1}{2}\pi - \epsilon,$$

where ϵ is a small angle; in absence of the disturbing mass-point $\Re\epsilon$ would clearly be the distance from the polar plane (which is also the largest possible sphere) measured along a straight perpendicular to it, all such perpendiculars crossing in O, and in no other point. The behaviour of g_1 can be seen directly from (58), which gives, for any ϵ,

$$-\frac{1}{g_1} = 1 - \frac{2L/\Re}{\cos\epsilon \sin^2\epsilon},$$

and therefore, at the polar itself, $\dfrac{1}{g_1} = +\infty$, $g_1 = 0$, while just above that sinister plane g_1 is a small fraction and, worst of all, a *positive* one. This means that the 'natural' length of a radial element $\sqrt{-g_1 dr^2}$ would be *nil* at, and *imaginary* just before reaching, the polar. Nay, even before attaining it,

* As H. Weyl likes to call it.

we would be stopped by a formidable barrier placed (approximately) at a distance $\Re\epsilon = 2L$ from the polar, e.g. 3 kilometres from the sun's polar. For here g_1 jumps from $-\infty$ to $+\infty$, to run then down to nil at the polar itself. Thus, a radially oriented rod (unlike a transversal one which would behave regularly) would show at and near that plane the most extravagant singularities. And that these cannot be 'transformed away' (as the relativists are wont to say) by introducing other, say 'isotropic', coordinates, can readily be shown.

Equally unsatisfactory and not less fantastic is the behaviour of the component g_4 of the metrical tensor. Take only the polar itself. Then ϵ can be assumed small even when compared with $L/\Re$, so that by the first and third of (58), $g_4 = \dfrac{\Re^2 \epsilon^2}{16L^2}$. This expression certainly holds good down to $\epsilon = 0$. Thus we have at the polar itself $g_4 = 0$, which is bad enough. So much so, in fact, that it scarcely calls for any further comments.

That this extravagant behaviour of g_1 and g_4 at the polar of our mass-centre (of every 'mass-centre', in fact, whether it stands for a sun or a microscopic dust particle) is not a mere mathematical illusion, that it cannot be transformed away, can best be seen by evaluating the curvature *invariant* R, since this is independent of the choice of the coordinates. Now, in general, $R = \kappa T + 4/\Re^2$, and in our case

$T = \rho_0 - 3p$, while (a') with $p_0 = 0$,* i.e. $A = \rho_0$, gives $(p+\rho_0)\sqrt{g_4} = \rho_0$, so that

$$R = \frac{6}{\Re^2}(2 - g_4^{-\frac{1}{2}}).$$

Now, at the polar $g_4 = 0$. Thus, while near the mass-centre, say in planetary regions of our sun, the world-curvature, $R/6 \doteqdot (1-L/r)\Re^{-2}$, is but a little below its normal value $1/\Re^2$, it becomes at the polar negatively infinite, to wit, as $-1/\epsilon$. If my readers desire perforce a two-dimensional analogy, they can compare this unfortunate 'horizon' with a razor-sharp ridge or, perhaps, crease, on a generally smooth surface.

In fine, Einstein's cylindrical world, even when helped up by that *deux ex machina*, the universal pressure p, gives for the planetary neighbourhood, of our own sun itself, results which to all purposes are indiscernible from his older, homaloidal space-time, and at the sinister 'horizon', the polar of the sun (or of its centre), it leads to singularities so shocking and unlike anything we know from physics or astronomical observation, as to make that cylindrical cosmology decidedly untenable.† Nor does Einstein

* If we assumed a non-vanishing residual pressure, $p_0 \neq 0$, the final result would show the same singularity.

† One might still urge that, as nobody has actually explored that sinister neighbourhood of our sun's polar (which is too distant even for our best telescopes), the difficulties pointed out above are harmless or 'of a purely academic nature', so to speak. Yet, since near that polar there are, especially in view of Einstein's belief

himself seem to insist upon it, these eight years or so, although he never as much as hinted, publicly, at the need of abandoning it.

But it is time to close this lengthy discourse on the cylindrical world. With due respect to the founder of modern Relativity, this superstructure of his masterly theory is entirely indefensible.

As a matter of fact, during the last few years nobody has seriously adhered to this cosmology or the kind of metrical spacetime implied in it—with, however, one exception, and a non-uninteresting one too. I have in mind the last section of Dr. Hubble's 1926-paper on extragalactic nebulae. But as Part II seems already too long, this championship of the cylindrical world will better be discussed in the next part of the book. This will not occupy very many pages. And it will be followed, in Part IV, by a careful discussion of the possibilities afforded by de Sitter's perfectly isotropic world.

in a roughly *uniform* distribution of galaxies over the elliptic space, other suns, stars, or nebulae, and since *their* polars pass through our very vicinity, the region occupied by our own solar system would be full of such surfaces, or rather zones, of singular behaviour of the metrical tensor, leading to the most extravagant catastrophic happenings among the members of the solar system. Each of these polars would to all purposes act as some super-adamantine wall. In fact, by e.g. (a') above, $g_4 = 0$ implies an infinite pressure p. Nor can any of these singularities be got rid of by transforming the coordinates. In fine, they are intrinsic.

PART III

DR. HUBBLE'S SIZE-ESTIMATE OF EINSTEIN'S CYLINDRICAL SPACETIME

THE purpose of this, rather short, part of the book is to give an account of Dr. Edwin Hubble's recent attempt not only to draw a bold picture of the whole universe of celestial bodies, but also to determine the radius of the elliptic space (section of cylindrical world) throughout which all these glories of heaven, the galaxies, are equably distributed.

But before taking up this subject, it may be well to give here some idea of the size and population of our own galaxy, the familiar Milky Way. It is true that most of my readers know everything (that has been found out) about this, so to speak, home-universe of ours. Yet some others might not have had the opportunity of getting quite the latest and most reliable information about this stellar system. And for the sake of these a page or so of a purely descriptive nature may well be inserted here.

It is surprising how much the dimensions of our Galaxy have grown within the last score of years—not in reality, of course, but in the opinion of the leading astronomers and astrophysicists. In fact, while, even in 1921, H. D. Curtis still tenaciously defended the views held ten to twenty years earlier

by Newcomb, Charlier, and other leaders in stellar astronomy, and ascribed to the galaxy a diameter not exceeding 30,000 light-years, in our days Shapley's more recent estimate,* which sets 300,000 light-years as its lowest limit, is generally accepted. Such is the diameter in the galactic plane. That perpendicular to it is estimated to be about ten times smaller, the whole assemblage of stars being, roughly, of the shape of a rotational ellipsoid, symmetrical around the shorter axis. Our solar system is placed at a distance of about one-sixth of the equatorial semi-axis from the centre of this gigantic ellipsoid.

The total number of stars composing it is, according to Chapman and Melotte, not less than 10^9 and cannot be much greater than 2.10^9, the average mass of a star being $\frac{1}{3}$ of that of our sun or somewhat less. (This information I derive from Eddington's *Stellar Movements*, &c., London, 1914.) According to Kapteyn's estimate, accepted by leading astronomers even ten years ago, the total mass of our galaxy is $\frac{1}{3}\,10^{10}$ suns.† We may as well mention that the latter mass-unit, much in use, is, in round figures, 2.10^{33} grammes.‡ In certain respects, then,

* Cf. H. Shapley and H. D. Curtis on 'The Scale of the Universe', *Bull. Nat. Res. Council*, No. 11, Washington, 1921.

† To which corresponds the gravitation radius

$$L = \frac{1{\cdot}5}{3}\,10^{10} = \tfrac{1}{2}\,10^{10}\ \text{km.},$$

or only about 33 astronomical units, or 1/6000 of a parsec.

‡ More accurately, $1{\cdot}985.10^{33}$.

our Milky Way can be imagined as a rotational ellipsoid with semi-axes

$$a, b = 150{,}000, \text{ and } 15{,}000 \text{ light years}$$

or

$$46{,}000, \text{ and } 4{,}600 \text{ parsecs},$$

having the mass $\frac{2}{3}10^{43}$ grammes. This gives for the mean mass-density of our galaxy $\rho_g = 8 \,.\, 10^{-5}$ suns per cubic parsec. (The density, always of *visible* stars only, in the neighbourhood of our solar system is estimated at $\frac{1}{50}$ sun per parsec^3.)

At present, however, we wished mainly to recall the geometrical dimensions (a, b) of the galactic system. Its mass will be of interest presently.

Since the greatest distance apart of two points possible in elliptic space is half the total length of a straight line, $\frac{1}{2}\pi\mathfrak{R}$, the curvature radius of our space, if this be elliptic (polar) must certainly be greater than

$$\frac{2}{\pi} 92{,}000$$

or, in round figures,

$$\mathfrak{R} > 6 \,.\, 10^4 \text{ parsec, i.e. } \mathfrak{R} > 1{\cdot}2 \,.\, 10^9 \text{ astr. units.}$$

This, at any rate, is a lower limit for the radius, and although we do not doubt an instant the existence of extragalactic masses, and huge ones too, this lower limit is not uninteresting as a preliminary piece of information, and as such is worthy of a moment's consideration. Suppose, however, we closed our eyes to the demonstrated inadmissibility of Einstein's

cosmos (i.e. to those fatal polars of all mass-centres). Then we should have to claim 'room enough' to lodge at least all that mass of our own galaxy, i.e., by (50),

$$\mathfrak{R} > \frac{4}{\pi} \cdot \tfrac{1}{3} 10^{10} \frac{\text{suns}}{c^2},$$

and since 1 sun$/c^2$ = 1·47 km.,

$$\mathfrak{R} > 6{\cdot}2 \,.\, 10^9 \text{ km.},$$

which, fortunately, is amply harmonizing with the last figure, $\mathfrak{R} > 1{\cdot}8 \,.\, 10^{17}$ km. This, however, is but a cheap triumph.

In fact, outside our galaxy there is undoubtedly a multitude of stellar systems, Dr. Curtis's 'island universes', represented in his opinion (which is contested by Shapley and others) by the spiral nebulae; also, possibly, a huge number of non-nucleated gaseous nebulae; all of these, perhaps, as massive as, or even more than, our home-universe. In view of this, and accepting provisionally $5 \,.\, 10^5$ light years or $5 \,.\, 10^{18}$ km. (Curtis's lower estimate) as the distance apart of these 'island universes', one could scarcely avoid making $\mathfrak{R}$ some thousand times bigger than the limit derived from our own galaxy, i.e. $\mathfrak{R} \doteqdot 10^{12}$ astr. units or, always in round figures, $\mathfrak{R} \doteqdot 10^{20}$ km. This, however, would mean, by (50), a total mass having a gravitation radius $L = \frac{\pi}{4} 10^{20}$ km. But to

this huge total our sun contributes only $1\frac{1}{2}$ km., and our whole galaxy hardly more than 10^{10} km. One would thus require 10^{10} such galaxies. But to accommodate these, spaced as just mentioned, the last-said space ($\pi^2\ 10^{36}$ cubic astr. unit) would not be ample enough. Still it would be childish to deny the possibility of a much greater curvature radius and of many more galaxies—and so on.

One sees at a glance that Einstein's quaint, though at first promising, equation (50) is essentially elusive. In fact, neither its first nor its second member (M and $\mathfrak{R}$) are ascertainable as to their approximate values or only their upper limits. If $\mathfrak{R}$, say, could be found independently of that formula, the case would be different. But it cannot thus be found; none of the observable implications of Einstein's cosmology contain $\mathfrak{R}$.*

* In fact, as we saw in Part II, Einstein's parallax formula does not sensibly differ from the classical one, that corresponding to $\mathfrak{R} = \infty$. To see how nearly this approximation holds develop (52′) up to the fifth power of $r/\mathfrak{R}$ and notice that since, at any rate, only small parallaxes come into play, tan p can be replaced by p. Then the result will be

$$r = \frac{a}{p}\left(1 - \frac{a^2}{3p^2\mathfrak{R}^2}\right),$$

which, indeed, is hopelessly indistinguishable from $r = a/p$, corresponding to $\mathfrak{R} = \infty$. For even if $p = 0\cdot''001 = \frac{1}{2}10^{-5}.10^{-3} = \frac{1}{2}10^{-8}$ (actually no parallax smaller than about $0\cdot''02$ can be determined by the trigonometric method), the second term is $\frac{4}{3}10^{16}\left(\frac{a}{\mathfrak{R}}\right)^2$, and since $\mathfrak{R}$ is certainly greater than, say, $10^{11}a$, this fraction is smaller than $\frac{4}{3}10^{-6}$. Again, turning to planetary motion, it can readily be shown (cf. *Theory of Relativity*, p. 494) that the effect due to a finite curvature

Such being the case, it would seem futile to apply this formula to no matter how rich astronomical data, even apart from the difficulty represented by those mass-centre polars. Yet Dr. Hubble has found it worth while to attempt it. And, as promised, we will now give an abbreviated account of his investigation as described in his paper on 'Extra-Galactic Nebulae' (*Contr. Mount Wilson Obs.*, No. 324).*

Hubble's contribution is the result of a statistical investigation of 400 *extra-galactic nebulae.*

For these nebulae the total visual magnitudes have been determined by Holetschek (1907).†

radius upon the secular perihelion motion of a planet bears to the famous Einsteinian effect in a homaloidal spacetime,

$$\delta\varpi_0 = \frac{24\pi^3 a^2}{c^2 T^2 (1-\epsilon^2)},$$

the ratio

$$\frac{\delta\varpi - \delta\varpi_0}{\delta\varpi_0} = \frac{1}{12}\left(1 - \frac{7}{3}\epsilon^2\right)\left(\frac{cT}{\pi\mathfrak{R}}\right)^2,$$

which is, essentially, somewhat smaller than the squared ratio of the light-year of the planet (cT) to the total length ($\pi\mathfrak{R}$) of a straight line in elliptic space. Is it necessary to say how hopelessly small this is? Even if T were such as the period of Neptune, and $\mathfrak{R} = 10^{12}$ astr. units, this ratio would not mount much above 10^{-13}. Still less would the well-known ray-deflexion and gravitational spectrum-shift formulae of Einstein's older theory be affected by his newer cosmological speculations.

In fine, the only relation involving $\mathfrak{R}$ in a relevant way is (apart from the disastrous 'polars') $\mathfrak{R} = \frac{4}{\pi}\frac{M}{c^2}$, and even this, when pressed hard, turns out to be devoid of physical significance.

* Reprint from *Astrophys. Journal*, lxiv, pp. 321–69, 1926.

† For all references pertaining to Hubble's work the reader is naturally expected to consult his own paper, which does not lack bibliography.

Definite evidence as to distances (and dimensions) is, in Hubble's own words, 'restricted to six systems' (extra-galaxies). This, certainly, is an extremely meagre number. But it seems large enough to Dr. Hubble, whose faith in 'the general principle of the uniformity of nature' is so strong as to make him *infer*, without much ado, '*the similar nature of the countless fainter nebulae*', which he hastens also to accept as forming a homogeneous group.

It strikes one that Dr. Hubble has a rather strange conception of that great and valuable principle of *Uniformity of Nature.* Namely, he manifestly takes 'uniformity' to be a synonym of *Monotony* or homogeneity. Now, to use a phrase of Shakespeare's, 'Nature, my Goddess', is nothing less than monotonous. Just the contrary, it is infinitely varied, full of surprises, caprices, or what not, as, in fact, a true goddess ought to be. What is referred to as her Uniformity is not a yawning monotony of manifestations, but a certain permanence or invariance of (possibly only a very few of) her 'ways', deep traits of her character, commonly called Laws of Nature. Dr. Hubble, instead, uses that powerful word in the same sense as one would speak of the uniformity of a good many (newer) streets in London, consisting of hundreds of houses, all alike in size, shape, colour, and what not, an interminable chain of things hopelessly devoid of individualism, the delight of the economical building contractor, no doubt, but the horror of every one else. Fortunately, Nature does not build her works on this pattern. But is it really necessary to explain that the extremely fruitful Principle of Uniformity is utterly distinct from what Dr. Hubble makes it? And

the paucity of galaxies (400, nay *six*) actually studied, which have driven him to this interpretation of the great maxim, does not improve the situation.

To realize the extent of naïveté implied in such a procedure, imagine for a moment that a statistician bases his assertion of the existence of a correlation between alcoholism and crime, say, in London, upon just 400 merry-making individuals, of whom, moreover, only six were demonstrably criminal, according to definite evidence. Who would dream of building upon such a correlation? And notice that taking '400' and 'six', we have taken much too many. For while the population of (greater) London is, say 8.10^6 (of whom a large fraction, children, &c., have to be discarded), that of the skies, with galaxy as individual, is very likely much greater. But let us proceed, with Dr. Hubble as our guide.

Skipping his 'Part I', devoted to the classification of nebulae (of which the extra-galactic and 'regular' ones are subdivided into elliptical ones and spirals, while the 'irregular' ones are too faint to be classified at all), let us pass on to the next Part, apportioned to a statistical study of extra-galactic nebulae. Here we learn that from Hardcastle's list of nebulae (1914), the most homogeneous for statistical study, and containing all nebulae appearing in the Franklin-Adams charts, only 700 may be considered as decidedly extra-galactic. Thus far as to their mere numerosity. But when it comes to knowing their type, total

(visual) magnitude, and diameter, only about 400 extra-galactic nebulae are left over for the sake of world-building study, as was already mentioned. The various types of these galaxies are uniformly distributed over the sky, show similar spectra, and are endowed with radial velocities of, roughly, the same order. Now, since these features, jointly with the equality of the mean magnitudes and the uniform frequency distribution of magnitudes, are all consistent with the *hypothesis* that also *the distances and absolute luminosities are of the same order for the various types*, Dr. Hubble hastens to accept this 'sameness',* an 'assumption of considerable importance', though—as he does not fail to admit—one which 'unfortunately cannot yet be subjected to positive and definite tests'.

We then learn that the nebulae of each type show a linear relation between total magnitude and logarithm of diameter, say,

$$m_T = C - K \log d,$$

where the constant K is common to all types studied, whereas C varies progressively throughout their sequence.

We will take this correlation for granted, although the four graphs 'showing it' (Figs. 2 to 5), and especially the last two, do not exactly impress one as striking evidence.

* 'Sameness, equality, same general order' seem to be, for good or evil, Dr. Hubble's torch-bearers through the enigmatic abysses of the heavens.

Dr. Hubble then proceeds to reduce his nebulous material to a standard type, namely, by applying certain corrections to the original values of the constant C. This yields the much desired supreme uniformization, viz. a *single* formula,

$$m_T = 13{\cdot}0 - 5 \log d,$$

which, in Hubble's opinion, 'expresses the relation for all [extra-galactic] nebulae, from the Magellanic Clouds to the faintest that can be classified'. It certainly does 'express' it, if one so wills (cf. graph on p. 25, loc. cit.), only don't ask *how*. For every disinterested admirer of Nature the admission would be nothing less than painless. Said Heinrich Heine, in one of his songs on an intimate disappointment: 'Und ich hab' es doch getragen, Aber fragt mich nur nicht *wie*.' Let us then make the same sacrifice for the benefit of Dr. Hubble's barracks-universe of galaxies. Howsoever, the coefficient (*five*) of log d is agreeable to the familiar inverse-square law, and this does not fail to suggest to the celestial statistician that the extra-galactic host of nebulae are again 'all of *the same order of absolute luminosity*', so that their apparent magnitude becomes, of course, a ready measuring tap of their *distance*, from (about) the centre of our own galaxy, that is. With this, to say the least, very risky assumption Hubble's formula for the distance in parsecs (r, his D) becomes

$$\log r = 0{\cdot}2\, m_T + 4{\cdot}04, \quad . \quad . \quad . \quad (H_T)$$

m_T being the total apparent magnitude of a nebula. This gives, e.g., for the lesser and the greater Magellanic Clouds ($m_T = 0{\cdot}5$, $1{\cdot}5$), the distances $r = 13{,}800$ and 21,900 parsecs respectively,* and for the famous spiral Messier 33 (N.G.C. 598), for which $m_T = 7$,

$$r = 275{,}000 \text{ parsecs} = 890{,}000 \text{ light-years},$$

about the largest distance ever found for a celestial object. It may interest the reader to know that only three or four years earlier (*Mount Wilson Contr.*, No. 250, 1922) Dr. Hubble, using another method, has placed this spiral at a distance of only 31,300 parsecs, i.e. about nine times nearer. But this by the way only.

Hubble then passes to consider the masses of the extra-galactic nebulae. Of the two existing methods of estimating them that based on spectroscopic rotations involves the distance, which, however, is known with comparative accuracy only for *one* object, the spiral M. 31. There remains, therefore, only the other method, due to Öpik (1922), based on the assumption that the emissivity of the luminous material in spiral nebulae is about equal to that of our own galactic system. Öpik's formula is (with the sun's luminosity as unit)

$$\text{Mass} = 2{\cdot}6 \times \text{luminosity}.$$

A direct test of this formula is, for the aforesaid reasons, possible only in the case of Messier 31. Now,

* Shapley's distance estimates were (at least up to 1925) 25,000 and 35,000 parsecs.

for the mass of this object the spectrographic-rotation method gives $3{\cdot}5 . 10^9$, and Öpik's formula, $1{\cdot}6 . 10^9$ suns. These results are, no doubt, 'of the same order', as claimed by Hubble. Yet one is scarcely impressed by the reliability of the empirical formula, based as it is on very flimsy foundations.

Still less does one feel convinced by the passage following directly after that extremely meagre test (loc. cit., p. 43). This is so inexplicably bold as to deserve a verbal quotation:

'Öpik's method leads to mass-values that are reasonable and fairly consistent with those [plural!] obtained by the independent spectrographic method. Therefore, in the absence of other resources, its use for deriving mass of the normal [average] nebula appears to be permissible. The result,

$$2{\cdot}6 . 10^8 \text{ suns},$$

corresponding to $-15{\cdot}2$ as the absolute magnitude [of an average nebula] is probably of the right order.'

Commentaries on this summary settling of the average mass of an extra-galactic nebula are scarcely necessary.

With equal ease Dr. Hubble now proclaims the spacing of nebulae to *be approximately uniform*, and finds for its density the value

$$\rho = 9 . 10^{-18} \text{ nebula per cubic parsec},$$

i.e. with the last-quoted mass,

$$\rho = 2{\cdot}3 . 10^{-9} \text{ suns per cubic parsec},$$

or in c.g.s. units,

$$\rho = 1{\cdot}5 . 10^{-31}.$$

He adds, cautiously enough, that 'this must be considered as a *lower limit*, for loose material scattered between the systems is entirely ignored'.

Notice that this density deserves, in the best case, to be considered only as the mean density of mass-distribution in the region of space, say V_1, over which the actually observed nebulae are scattered, but certainly not over the whole volume V of space (whatever its nature and extension). Yet, as we will see in a moment, Hubble assumes $\rho = \frac{3}{2}10^{-31}$ to be *the universal density* of mass-distribution. Now, even if we agreed with him that the nebulae are more or less uniformly distributed over V_1, how can we know that this region does not form only a small fraction of the whole space (if this be finite) in which, say, other such regions V_2, V_3, &c., of the order of V_1 and perhaps similarly populated, are distributed, in distances apart comparable with or much greater than the dimensions of those galaxies of galaxies placed in V_1, V_2, V_3, and so on. In this manner (as e.g. in Charlier's hierarchic cosmology) every desired value of ρ, down to zero itself, can be obtained, without clashing with the density actually found in V_1. Thus $\rho = 1{\cdot}5 . 10^{-31}$ has no claims whatever to be looked upon as the '*lower limit*' of mean (universal) density; it is just a regional density, ρ_1. Even if millions of millions (instead of just 6 or 400) nebulae

were actually observed and measured, and found to be very evenly distributed, no lower limit of density ρ other than zero could be inferred.

We come at length to Dr. Hubble's final stroke, with which he crowns his laborious inquiry in the last section of the paper. It covers only about half a page and is entitled 'The Finite Universe of General Relativity'.

Without much ado Dr. Hubble transcribes Einstein's formula, (49) above, i.e. (writing ρ instead of our ρ_0)

$$\mathfrak{R} = \frac{c}{\sqrt{4\pi k\rho}}, \quad . \quad . \quad . \quad . \quad (49')$$

from Prof. Haas's *Theoretical Physics*, a semi-popular book which I had no opportunity to peruse but which probably does not contain any such criticism as that derivable from the 'polar' of every mass-centre. Nor does Hubble seem to care for any inquiry as to the legitimacy of this apparently simple formula. He is content to introduce the subject by the few words: 'For the present the simplified (?) equations which Einstein has derived for a spherically curved space can be used.' He then writes down the above (49′), and (for the volume and mass-total) $V = 2\pi^2\mathfrak{R}^3$, $M = \frac{\pi c^2}{2k}\mathfrak{R}$, of which the latter, as we know, is only ρ (49′) multiplied by V. Well understood, Hubble accepts, after Einstein, an antipodal space. For the polar one, or 'elliptic' proper, $V = \pi^2\mathfrak{R}^3$, and there-

fore $M = \dfrac{\pi c^2}{4k}\Re$, and if M is in astronomical units,

$M = \dfrac{\pi c^2}{4}\Re$, as we saw in Part II.

Substituting in (49′) the above-quoted mean density $\rho = 1{\cdot}5\,.\,10^{-31}$, Hubble finds $\Re = 8{\cdot}5\,.\,10^{28}$ cm. or

$$\Re = 2{\cdot}7\,.\,10^{10} \text{ parsecs.} \quad . \quad . \quad . \qquad \text{(E.H.)}$$

The corresponding mass-total of the universe is, for antipodal space, $M = 9\,.\,10^{22}$ suns $= 3{\cdot}5\,.\,10^{15}$ normal nebulae, and for polar space, one half of this.

He concludes his paper by recalling that the distance up to which a 100-inch reflector would detect a 'normal' nebula, i.e. $4{\cdot}4\,.\,10^7$ parsecs, amounts to as much as 1/600 of his $\Re$. And since unusually bright nebulae, as Messier 31, could be photographed at several times that distance, it may—with increased plate speed (photographic sensitivity) and telescopic sizes—soon become possible to observe 'an appreciable fraction of the Einstein universe'.

Unfortunately, however, and even quite apart from the very weak support of Hubble's ρ-value, *no part* of 'Einstein's universe' can ever be observed, since this (cylindrical) universe, leading to absurd consequences, cannot—in any reasonable sense of the word—be said to exist at all.

And, even if one closed one's eyes * to the essential

* But why *should* one?

inadmissibility of the cylindrical spacetime, the formula

$$M = \frac{\pi c^2}{4} \mathfrak{R},$$

i.e. Hubble's

$$4{\cdot}5 \,.\, 10^{22}\ \text{suns} = \frac{\pi c^2}{4}\, 2{\cdot}7 \,.\, 10^{10}\ \text{parsecs},$$

cannot, independently, be tested. It, actually, has no physical (phenomenal) content, as was already pointed out before. Even if the M-value (or ρ-value) were legitimately determined (which it certainly is not), '$\mathfrak{R} = 2{\cdot}7 \,.\, 10^{10}$ parsecs' would be devoid of physical meaning. For in Einstein's cylindrical cosmology there is not a single other formula into which we might substitute this $\mathfrak{R}$-value and make with its aid an experimentally verifiable prediction. Notice that this inanity or irrelevance is by no means an exclusive feature of the Einsteinian world radius. The case of 'the diameter of an electron' (say $2 \,.\, 10^{-13}$ cm.) is exactly similar.* Curiously enough, the similarity of these two cases of physical irrelevance of a magnitude is much more intimate than one might at first suppose. In fact, a definite value of the diameter of an electron, say the familiar $2 \,.\, 10^{-13}$ cm., is derivable from its (well-known) total charge only through the perfectly arbitrary assumption of its space-distribution, say homogeneous.†

* For the time being, at least.

† It may be mentioned here that some years ago I proposed to consider a charge-distribution within the electron of the type

And so also has that big $\mathfrak{R}$ of the world been derived by Hubble with the aid of the ill-founded acceptance of a uniform distribution of nebulae throughout the space.

But enough has now been said about this piece of astro-statistical work to convince the reader of the illegitimacy of its results. Nor will there be any need for returning to Einstein's cylindrical spacetime, sounded for its size by Dr. Hubble.

$\rho = \rho_0 e^{-r/\lambda}$, $\lambda =$ const., which would imply an infinite 'diameter' of the electron, without clashing with any known fact.

PART IV

PHYSICAL PROPERTIES OF ISOTROPIC SPACETIME. CORRELATION BETWEEN DISTANCE AND RADIAL VELOCITY

WE saw in Part II that de Sitter's spacetime at large, satisfying Einstein's amplified gravitational equations, and characterized by the line-element

$$\left.\begin{aligned} ds^2 &= \cos^2\sigma\,.\,c^2dt^2 - dl^2, \quad \sigma = r/\mathfrak{R}, \\ dl^2 &= dr^2 + \mathfrak{R}^2\sin^2\sigma\,[d\phi^2 + \sin^2\phi\,d\theta^2], \end{aligned}\right\} \qquad (59)$$

is perfectly *isotropic* and, *eo ipso*, homogeneous as well. Its curvature invariant $R = g^{\iota\kappa}R_{\iota\kappa}$ bears to the curvature radius $\mathfrak{R}$ of the elliptical three-space (a section of this fourfold normal to the t-axis) the simple relation

$$R = \frac{12}{\mathfrak{R}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (60)$$

so that the radius $\mathfrak{R}$ also is a four-invariant with respect to any transformations of the coordinates. This line-element or the corresponding tensor $g_{\iota\kappa}$ is a solution of the said field equations for $T_{\iota\kappa} = 0$ and $\lambda = \frac{1}{4}R = 3/\mathfrak{R}^2$, i.e., ultimately, of the equations

$$R_{\iota\kappa} = \frac{3}{\mathfrak{R}^2}g_{\iota\kappa}. \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

It is, of course, but a special solution of these differential equations, corresponding to a complete absence of matter. A more general solution, covering the case of a mass-centre, say of mass c^2L, placed

at $r = 0$ is easily found. (Cf. *Theory of Relativity*, p. 507.) With the coordination

$$x_1, x_2, x_3, x_4 = r, \phi, \theta, ct$$

it is

$$g_1 = -\frac{\cos^2 \sigma}{g_4}, \quad g_2 = g_3/\sin^2 \phi = -\mathfrak{R}^2 \sin^2 \sigma,$$

$$g_4 = \cos^2 \sigma - \frac{2L}{\mathfrak{R} \sin \sigma}. \qquad (62)$$

For $L = 0$ it reduces, of course, to (59).

We shall mainly be concerned with the latter, more special, solution and its implications. Yet, before discussing and applying these, it may be well to convince ourselves that, unlike Einstein's cylindrical space, the polar of the mass-centre is by no means 'fatal' to this cosmology, showing in fact no physical peculiarity at all.

In absence of a mass-centre g_4 vanishes at the polar, $\sigma = \frac{1}{2}\pi$, which, however, can readily be shown not to lead to any trouble. This can best be seen by remembering that *any* plane of the elliptic space can be made to be the plane or greatest sphere $\sigma = \text{const.} = \frac{1}{2}\pi$, and, accordingly, any point of the (perfectly homogeneous) manifold, the origin $\sigma = 0$, i.e. the pole of that surface, or its centre. Nor does the invariant R acquire at the polar, for any chosen origin, a value differing from $12/\mathfrak{R}^2$, which it has throughout the manifold. Now, in the presence of a mass-centre Lc^2, the apparent or merely analytical peculiarities at and near its polar are materially the

same as in the absence of such a centre. The only difference is that, instead of at $\sigma = \pi/2$, g_4 vanishes at a slightly smaller distance, $\sigma = \frac{\pi}{2} - \tau_0$, where τ_0 is the smallest root of $\sin^2 \tau \cos \tau = 2L/\mathfrak{R}$ [this by the last of (62)] or, approximately,

$$\tau_0 = \left(\frac{2L}{\mathfrak{R}}\right)^{\frac{1}{2}} \left(1 + \frac{5L}{6\mathfrak{R}}\right).$$

At the polar itself we have $g_4 = -2L/\mathfrak{R}$ which, in the case of a mass such as the sun or even our whole galaxy, is a very small negative fraction. In fine, $\sqrt{g_4}\,dt$ vanishes at a distance $\mathfrak{R}\tau_0$ from the polar and becomes imaginary between this sphere and the polar of the mass-centre. The coefficient, $-\cos^2 \sigma/g_4$, of dr^2 is precisely as in the cylindrical world, and has therefore the same analytical singularity at the polar. But there is, fortunately, this difference, that in the present case the curvature invariant R remains regular. Nay, it has, by (62), the value $R = 12/\mathfrak{R}^2$ at the polar itself as anywhere else—which can readily be verified. Nor are there any other intrinsic * or physical singularities. This agreeable behaviour is due to the circumstance that a point-mass in this isotropic spacetime does not call for an auxiliary pressure which in Einstein's cylindrical world gave rise to intolerable physical singularities at and near the polar.

Let us now return to the element (59), holding in

* i.e. such as cannot be transformed away.

de Sitter's spacetime devoid of masses (and energy), except, of course, the conceptual test-particle whose mass will be taken as negligible. But before turning to the usual questions, of light-lines, geodesics, or free particle motion with some of their most interesting implications, we cannot help recalling once more that the perfect isotropy of the metrical spacetime determined by this line-element is best expressed, *ad oculos*, so to speak, by the values of the (only surviving) components of the corresponding mixed contracted curvature tensor, which are in any coordinate system,

$$R_1^1 = R_2^2 = R_3^3 = R_4^4 = \frac{3}{\mathfrak{R}^2} = \text{const.},$$

as in (34 a) above.

Needless to repeat, *all* properties of this *empty* isotropic world are derivable from (59) together with the interpretation principles: $ds = 0$ for light propagation, $\delta\int ds = 0$ for free motion.

The elliptic space whose element is dl, being assumed to be of the *polar* kind, it will be all covered by the interval $\sigma = 0$ to $\pi/2$. The origin O is, as has already been pointed out, any point, and so also is its absolute polar, say ω, a plane as good as any other in that space, in spite of all analytical appearances ($g_4 = \cos^2\frac{\pi}{2} = 0$, &c.). More than this, we know beforehand that all points of the *four*-fold, which is homogeneous owing to its very isotropy, are completely equivalent to each other.

Whence we can see at once that, no matter how strange the findings, chronometrical or geometrical, optical and kinetic, of an observer exploring this world from the station O, they will all be exactly the same for one stationed elsewhere, say at O'. Not that O' with its neighbourhood need appear to the former observer as O appears to him while he stays there. But, whatever the distinctive features, they will reappear if this observer moves (in space and time) to O' and looks back to his original station.

Keeping this in mind, however, it will be formally convenient to imagine all statements to refer to an observer placed at the origin O ($\sigma = 0$) of the coordinates, unless otherwise stated.

The space-section characterized by the term $dl^2 = \mathfrak{R}^2 \sin^2 \sigma \, (d\phi^2 + \sin^2 \phi \, d\theta^2)$ of the full line-element ds^2 offers in itself no peculiarities whatever. A measuring rod will have the same 'natural' length in every orientation and position, and this will thus be identical with its system-length. The entire novelty of this empty spacetime of de Sitter is contained in the first term, $\cos^2 \sigma . c^2 dt^2$, the system-time t (or its element) appearing with a function of distance, $r = \mathfrak{R} \sigma$, as a factor. And this factor, moreover, cannot be removed without spoiling, that is, complicating, the simple space-geometry.* In fine, the line-element of the isotropic world differs from

* An instructive example, originally due to de Sitter, will be found in my *Relativity*, under (27_1), p. 504.

that of Einstein's cylindrical four fold only by $g_4 = \cos^2 \sigma$ as against $g_4 = 1$. (For $\mathfrak{R} = \infty$ both, of course, would be dreary homaloids, as Clifford would put it.)

Thus the *proper time* $d\tau = ds/c$ of a clock, kept forcibly *at rest* at a distance r from the observer (at O), will be

$$d\tau = dt \,.\, \cos \sigma = dt \,.\, \cos (r/\mathfrak{R}),$$

and conversely, the system-time, which coincides with the proper or simply *the* time of the observer, will be

$$dt = d\tau \sec \sigma. \quad . \quad . \quad . \quad . \quad (63)$$

In words, if the clock be an ideal one, such perhaps as a hydrogen atom in a standardized state (electron on one-quantic orbit, say), it will appear to the observer the slower the farther away from him it is placed, and will cease to go *for him*, when it is just at the observer's polar, $\sigma = \frac{1}{2}\pi$. There, in fact, everything would, by (63), and as most relativists somewhat over-hastily assert, be at a standstill, for our observer. The clause, or hint at usual over-hastiness, will be understood in the sequel when the unavoidable light-signalling of distant events is taken into account.

As has already been said, the vanishing of g_4 all over the polar is but an apparent singularity. But, though there is nothing physically singular to stumble against at the polar as at any other place (worldpoint) taken in itself, the relations implied in

formula (63), for a *pair of distant stations*, are by no means unreal; that is to say, they represent some verifiable physical facts related to any such pair of points, one occupied by a clock or anything equivalent, and the other by an observer or, perhaps, an automatically working spectrograph.

One such, predictable, phenomenon is de Sitter's well-known red-shift of the spectral lines of remote celestial bodies as a pure distance effect. In fact, if the much debated permanence of atoms as clocks be assumed to hold good, a reasoning similar to that by which Einstein has, some fifteen years ago, deduced his gravitational red-shift,* leads to a spectrum shift which is expressed by the formula

$$\frac{\lambda}{\lambda_0} = \frac{dt}{d\tau},$$

where λ and λ_0 are the wave-lengths of a definite line in the stellar and the terrestrial spectrum. This gives, by (63), and with $\delta\lambda$ written for $\lambda - \lambda_0$ and, in the denominator, λ for λ_0 (which means neglecting squares and higher powers of the small fraction $\delta\lambda/\lambda_0$),

$$\frac{\delta\lambda}{\lambda} = \sec\sigma - 1 \div \tfrac{1}{2}\frac{r^2}{\mathfrak{R}^2}, \quad . \quad . \quad . \quad (64)$$

higher powers of $r/\mathfrak{R}$ being rejected. Such, then, should be the spectrum shift, a *second-order* distance effect, under the assumption (made explicitly by

* This can now be considered to be well verified for the sun and, still better, for the companion of Sirius.

de Sitter *) that the light source is at rest relatively to the observer.

Now, quite independently of the fact that a star does not, nay, in this isotropic world (as we shall see presently) cannot be at rest relatively to our observer, quite apart from this, the de Sitter effect (64), of second order in $r/\mathfrak{R}$, is much too small to be observed in general, and more especially on those celestial objects, principally the B-stars, which de Sitter had in view while attempting to test his formula. In fact, the average distance of the B-stars is generally taken to be of the order of $3 . 10^7$ astronomical units, whereas $\frac{\pi}{2}\mathfrak{R}$ is certainly greater than the distance of the nebula Messier 31 placed by Hubble at something like 925,000 light-years or about $6 . 10^9$ astr. units, so that

$$\mathfrak{R} > 4 . 10^9$$

and scarcely smaller than 10^{10} astr. units. Consequently, the de Sitter effect would for the B-stars be a good deal smaller than $5 . 10^{-6}$,† and much too small to be detected, especially as it would first have to be disentangled from a host of 'shifts' of an altogether different origin. We may mention that, according to de Sitter's quotation, these B-stars or helium stars show a systematic shift of lines towards the red end of

* *Monthly Notices, Roy. Astr. Society* for Nov. 1917.

† That is, smaller than what would correspond to a receding radial velocity of 1·5 km./sec.

the spectrum corresponding to a positive (receding) radial velocity of 4·5 km./sec. Of this de Sitter proposes to throw one-third upon Einstein's gravitational effect, due to the star's own field, and the remaining two-thirds upon that distance effect. This has led him to write, in astr. units,

$$\frac{1}{2}\left(\frac{3\,.\,10^7}{\mathfrak{R}}\right)^2 = \frac{\delta\lambda}{\lambda} = 10^{-5},$$

which gave

$$\mathfrak{R} = 6{\cdot}7\,.\,10^9 \text{ astr. units},$$

and this is inadmissibly small. There is, moreover, nothing to compel us to ascribe as much as two-thirds of the said systematic shift to the 'distance-effect', i.e. to the dwindling of g_{44} with the mere increase of distance. Nor does the marked preponderance of *positive* radial velocities among the spiral nebulae, agreeable to formula (64), help very much to strengthen the position of that interesting formula, the more so, as two or three of the best investigated spirals show large *negative* velocities (of the order of 300 km./sec.) and as we are entirely ignorant with regard to the behaviour of southern nebulae.

So much as to the direct implications of the structure of the time term of ds^2, or the form of g_{44} taken by itself.

Let us, in the next place, pass to consider the *geodesics* of the isotropic world, that is, the laws of motion of a free test particle inserted in this space-

time. These are most simply derived by using a Lagrangian development of $\delta\int ds = 0$ with (59) as line-element. In fact, the variation of the independent variables t and θ gives at once

$$\cos^2\sigma\,.\,c\dot{t} = k, \quad \Re^2\sin^2\sigma\,.\,\dot{\theta} = p,$$

where the dot stands for d/ds, and k, p are integration constants. A substitution of these two first integrals into (59) itself gives without trouble the required equations reducing the problem to mere quadratures,

$$\frac{d\theta}{dt} = \frac{cp}{k\Re^2}\cot^2\sigma \quad . \quad . \quad . \quad . \qquad (65)$$

$$\frac{1}{c}\frac{dr}{dt} = \pm\cos\sigma\left[1 - \frac{\cos^2\sigma}{k^2} - p^2\frac{\cot^2\sigma}{\Re^2 k^2}\right]^{\frac{1}{2}}. \qquad (66)$$

The orbit of the particle in the elliptic space is obtained by eliminating t between these two equations.

This orbit is, generally, by no means a straight line of the elliptic space. In fact, the differential equation of an elliptic straight is $\sin^2\sigma\,.\,d\theta/dl = \text{const.}$, to which the orbit (65), (66) reduces only in the very special case $p = 0$, that is to say, $\theta = \text{constant}$—when the orbit passes through the origin. But even then, as will be seen presently, the particle does *not* move uniformly. These are some of the characteristic differences between the isotropic spacetime and Einstein's cylindrical world.*

* That the orbit of a free particle should be straight or not according as it passes or does not pass through the origin sounds at first

But let us return to the general case of free motion. Since the variation of ϕ in $\delta\int ds = 0$ gives at once $\dot{\phi} = 0$, or $\phi = \text{const.}$, we may as well put $\phi = \pi/2$, so that for any world-geodesic,

$$ds^2 = \cos^2\sigma \,.\, c^2dt^2 - dr^2 - \Re^2 \sin^2\sigma \, d\theta^2 . \quad (67)$$

As to the space-section of this geodesic, i.e. the orbit of the particle in elliptic space, this will, according to (65), (66), be determined by

$$\frac{\cos^2\sigma}{\sin^4\sigma}\left(\frac{d\sigma}{d\theta}\right)^2 = \frac{\Re^2}{p^2}(k^2 - \cos^2\sigma) - \cot^2\sigma. \quad (68)$$

Put

$$u = \frac{1}{\sin\sigma}, \quad A = 1 + (k^2 - 1)\frac{\Re^2}{p^2}. \quad (69)$$

Then this equation will become

$$\left(\frac{du}{d\theta}\right)^2 = A - u^2 + \frac{\Re^2}{p^2u^2}, \quad (68')$$

whence the second-order equation

$$\frac{d^2u}{d\theta^2} + u + \frac{\Re^2}{p^2u^3} = 0,$$

of which the first two terms are perfectly familiar.*

very strange, especially as the origin O may be any point of the space. The puzzling impression disappears, however, when it is remembered that we are using a system-time which is just the *proper time* of the observer placed at O, that the worldline of this observer, $r = \text{const.} = 0$, is a world-geodesic, and that, therefore, all worldlines of free particles passing through O may be characterized intrinsically as forming a pencil of geodesics.

* For ever increasing $\Re$, u tends to $\Re/r$, so that for $\Re = \infty$ the last equation reduces to $d^2u/d\theta^2 + u = 0$, which then represents a straight line in Euclidean space, say, $r\cos\theta = \text{const.}$

But it is more convenient to use (68′), which calls only for a quadrature and gives, with $w = u^2$,

$$2\theta = \int \frac{dw}{\sqrt{\Re^2/p^2 + Aw - w^2}} = \text{arc sin}\,\frac{2w-A}{\sqrt{A^2+4\Re^2/p^2}}$$

or

$$\sin^2\sigma\,\{1+\sqrt{1+4\Re^2/p^2A^2}\,.\,\sin 2\theta\} = \frac{2}{A}.$$

Let θ_p be the value of θ at the polar of O, that is, for $\sigma = \pi/2$. Then the orbit equation can be written

$$\sin^2\sigma\left\{1+\frac{2-A}{A}\,\frac{\sin 2\theta}{\sin 2\theta_p}\right\} = \frac{2}{A}.$$

Thus, all orbits extend from $\theta = \theta_p$ to $\theta = \frac{1}{2}\pi - \theta_p$ and have the line $\theta = \frac{1}{4}\pi$ for their axis of symmetry. They resemble hyperbolae, though they do not, of course, deserve this name rigorously. A few such curves are drawn in Fig. 7, where points such as A and A' must be imagined to coincide, AOA' being a whole straight line. If θ be reckoned from the said line of symmetry, at which r clearly attains its minimum, say $r_0 = \Re\sigma_0$, the last equation becomes

$$\sin^2\sigma\left\{1+\frac{2-A}{A}\,\frac{\cos 2\theta}{\cos 2\theta_p}\right\} = \frac{2}{A}. \qquad (70)$$

The orbit extends from $-\theta_p$ to $+\theta_p$, this being determined by $\cos 2\theta_p = (2-A)/\sqrt{A^2+4\Re^2/p^2}$ or, in view of (69) and after some reductions, by

$$\tan\theta_p = \frac{\beta_0}{\sin\sigma_0}, \qquad (71)$$

where $\beta_0 = v_0/c$, v_0 being the velocity of the particle at the minimum distance, say the perihelion, if O

be the sun or its centre. Of course, $\sigma = \pi/2$, θ and $\sigma = \pi/2$, $\theta + \pi$ represent (for any θ) one and the same point of the (polar) elliptic space.

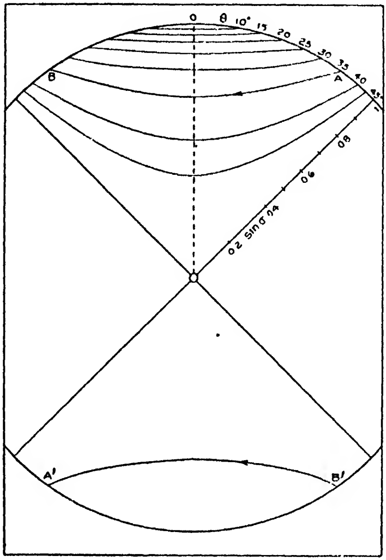

FIG. 7.

Let us still notice that, by (65) and (66), the square of the *resultant velocity* of the free particle,

$$v^2 \equiv c^2 \beta^2 = (dr/dt)^2 + \Re^2 \sin^2 \sigma \, (d\theta/dt)^2,$$

is, in the most general case,

$$\beta^2 = \cos^2 \sigma \, [1 - \cos^2 \sigma / k^2], \quad . \quad . \quad (72)$$

and in particular, for $k = 1$, $\beta = \frac{1}{2} \sin 2\sigma$, no matter whether the orbit passes through the origin or not.

Having thus settled the general case of free

motion, let us now consider at some length the purely *radial* motions along straight orbits, that is, pointing towards or away from the origin (commonly referred to as 'approaching' or 'negative', and 'receding' or 'positive'). By (66), with $p = 0$, we have for any such motion, referred to an origin which (as we already know) itself behaves as a free particle,

$$\frac{\Re}{c}\frac{d\sigma}{dt} = \pm\cos\sigma\sqrt{1-\cos^2\sigma/k^2}, \quad . \quad . \quad (73)$$

whence, first of all, we see that the motion is never uniform. The upper sign covers, of course, the case of a receding, and the lower that of an approaching particle. By differentiation we find for the (radial) acceleration, in *both* cases,

$$\frac{\Re^2}{c^2}\frac{d^2\sigma}{dt^2} = \tfrac{1}{2}\sin 2\sigma\left[\frac{2\cos^2\sigma}{k^2} - 1\right]. \quad . \quad (74)$$

Now, by the original meaning of the integration constant k,

$$\frac{1}{k^2} = \frac{1-\beta^2\sec^2\sigma}{\cos^2\sigma}, \qquad \beta = \frac{1}{c}\,|\,dr/dt\,|.$$

Two cases are, at this stage, to be distinguished.

First, the particle passes actually through the origin, with a velocity $c\beta_0$. Then

$$\frac{1}{k^2} = 1-\beta_0^2, \qquad k \geqslant 1. \quad . \quad . \quad . \quad (a)$$

Second, the particle reaches a minimum distance (perihelion) $r_0 = \Re\sigma_0$, where its motion is reversed. Then

$$k = \cos\sigma_0 \leqslant 1. \quad . \quad . \quad . \quad . \quad (b)$$

In accordance with (74) both cases of radial free motion are possible in de Sitter's spacetime, and both are of considerable interest.

From (73) and (74) we see, first of all, that the particle can be at permanent rest only at the origin or at the polar, $\sigma = 0$ or $\pi/2$. The latter situation is compatible with any value of k, and the particle will, from the O-point of view, in no finite time leave the polar. The case $\sigma = 0$ is compatible only with $k = 1$, that is, $\beta_0 = 0$ and $\sigma_0 = 0$. In fine, a free particle which is at rest at the origin will remain there for ever.

In the next place, consider a particle actually not at O, but one that would tend to O asymptotically, with $\beta_0 = 0$, or one that has left O at $t = -\infty$ with a vanishing velocity. Or, what practically amounts to the same thing, let β_0 or σ_0 be only small enough to make their squares negligible in the presence of 1. Then, by (a) or (b) respectively, $k = 1$, and equation (73) acquires the simple form

$$\frac{\Re}{c}\,\frac{d\,(2\sigma)}{dt} = \sin 2\sigma, \quad . \quad . \quad . \quad . \quad (73_0)$$

and (74),

$$\frac{\Re^2}{c^2}\,\frac{d^2\,(4\sigma)}{dt^2} = \sin 4\sigma. \quad . \quad . \quad . \quad . \quad (74_0)$$

The former equation is readily integrated. If t_m be an arbitrary constant, we have

$$\tan\sigma = e^{\pm\frac{c}{\Re}(t-t_m)}. \quad . \quad . \quad . \quad . \quad (73_0')$$

If the particle happens to recede from the observer,

it will do so for ever, *gathering speed* up to $t = t_m$,* when it will just reach the mid-point, $\sigma = \frac{1}{4}\pi$, between O and its polar, passing through it with the velocity $\frac{1}{2}c$. This, by the way, though huge, is still below the system-velocity of light at $\sigma = \frac{1}{4}\pi$, namely $c \cos\frac{\pi}{4} = c/\sqrt{2}$. Henceforth the particle will still recede but with *decreasing* velocity,

$$\beta = \tfrac{1}{2}\,\mathrm{sech}\,\frac{c}{\mathfrak{R}}(t - t_m),$$

which tends to zero, for $t = \infty$, at the polar. Thus the receding particle behaves at first, but up to $\sigma = \frac{1}{4}\pi$ only, as if it were *repelled* from O, and one may speak (with Eddington) of a 'scattering tendency' in de Sitter's world. But 'the general tendency to scatter' proclaimed by Eddington,† possibly under Weyl's inspiration, is a hasty and incorrect conclusion. For beyond the mid-point ($\pi/4$) the particle moves away more and more slowly. Again, if we take the equally probable case of an *approaching* free particle or, say, star, always disregarding gravitation, we will find that, once it has come between the mid-point and O, the star will tend towards the latter point with decreasing velocity, again as if

* In the first stage of motion, i.e. for small values of $ct/\mathfrak{R}$, and for small σ_0, e.g. (73$'_0$), re-written so as to make $\sigma = \sigma_0$ for $t = 0$, that is, $\tan\sigma = \tan\sigma_0\, e^{\pm ct/\mathfrak{R}}$, gives $r \doteq r_0 \pm c\,\sigma_0\, t$, that is to say, an approximately uniform motion, as might be expected.

† *Mathematical Theory of Relativity*, Cambridge Univ. Press, 1923, p. 161.

repelled. But let us trace its history back, nearer to the polar. There the velocity of the star towards the origin may even be almost evanescent, yet the star will continually gather speed towards O, as if attracted by this point, up to the huge value $\frac{1}{2}c$ which will be attained at the mid-point. A universal scattering tendency, as claimed by H. Weyl * and others is, then, by no means a characteristic feature of the isotropic spacetime and the corresponding cosmology, although restlessness of free particles is one. For all we know, every star or nebula answering the case of 'repulsion' may be matched by one fitting the case of 'attraction', the more so, as (in absence of g_{41}, g_{42}, g_{43}) every motion is reversible. Nay, if the distribution of stars or of galaxies of stars throughout the elliptic space is, or has been in the remote past, roughly uniform, there should be a much *greater* number of stars answering the second case, i. e. showing (always apart from gravitation) a gathering rather than a scattering tendency. In fact, the volume of a sphere of radius $\frac{1}{4}\pi\mathfrak{R}$, centred at O, is $V_1 = \pi^2\mathfrak{R}^3\left(\frac{1}{2} - \frac{1}{\pi}\right)$, and therefore the volume of the remaining elliptic space, $V_2 = \pi^2\mathfrak{R}^3\left(\frac{1}{2} + \frac{1}{\pi}\right)$, so that the ratio of the observer's own and the more distant regions of space is

$$V_1 : V_2 = \frac{\pi - 2}{\pi + 2}.$$

* *Raum-Zeit. Materie*, 5th ed., p. 322.

The latter is thus about five times more voluminous, and the celestial objects endowed with a gathering tendency should, therefore, be five times as numerous as those showing a tendency to scatter.

We will yet come back to the subject of the motion of a free particle in the empty world of de Sitter. Meanwhile, however, let us turn to consider the propagation of light or *the minimal lines*, $ds = 0$, of this spacetime. The differential equation of these lines, the light-lines, is obtained by equating to zero the line-element (59). Thus,

$$\frac{dl}{dt} = c \,.\, \cos\sigma, \quad . \quad . \quad . \quad . \qquad (75)$$

where dl is a line-element of the elliptic space, as given by the second of (59). If r be the radius of an expanding spherical wave centred at the source, which is placed at O, we have

$$\frac{dr}{dt} = c\cos\sigma \quad \text{or} \quad dt = \frac{\mathfrak{R}}{c}\frac{d\sigma}{\cos\sigma},$$

whence, the time of light-signalling from O to a sphere of radius $\mathfrak{R}\sigma$,

$$t = \frac{\mathfrak{R}}{c}\int_0^\sigma \frac{d\sigma}{\cos\sigma} = \frac{\mathfrak{R}}{c}\log\tan\left(\frac{\sigma}{2} + \frac{\pi}{4}\right). \quad . \qquad (76)$$

For $\sigma = \pi/2$ this becomes (logarithmically) infinite.

The shape of *the light rays*, in the three-space, follows most easily from the equation (68) of the orbit of a free particle, namely, as the limiting case

in which k and p are both infinite. Thus, and putting again

$$x = \Re \sin \sigma, \quad u = 1/x,$$

we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \text{const.},$$

whence

$$\frac{d^2u}{d\theta^2} + u = 0,$$

and the equation of a light ray becomes

$$\sin \sigma \,.\, \cos \theta = \text{const.} \quad . \quad . \quad . \quad (77)$$

This is by no means the equation of a straight line of the elliptic space (apart from the special case in which it passes through the origin). It would represent a straight in a space having the line-element $dl^2 = dx^2 + x^2\,d\theta^2$, while for elliptic space

$$dl^2 = dr^2 + \Re^2 \sin^2 \sigma \, d\theta^2 = \frac{dx^2}{1 - x^2/\Re^2} + x^2\, d\theta^2,$$

which is essentially different.

If we introduce, after de Sitter, a new variable h through

$$\sinh \frac{h}{\Re} = \tan \sigma, \quad . \quad . \quad . \quad . \quad (78)$$

then the line-element (59), for $\phi = \pi/2$, is transformed into

$$ds^2 = \operatorname{sech}^2\left(\frac{h}{\Re}\right) \cdot \left\{c^2\,dt^2 - dh^2 - \Re^2 \sinh^2 \frac{h}{\Re}\, d\theta^2\right\} \quad . \; (79)$$

Although the space of (79) is, of course, no more hyperbolic than that of (59),* and the path of free

* Against de Sitter's erroneous statement that 'the space of this

particles or light in it is not straight, yet the new variable h offers an advantage in treating optical problems. For since light propagation is expressed by $ds = 0$, we have, apart from the polar [where $\sinh(h/\Re) = \infty$],

$$c^2\,dt^2 = dh^2 + \Re^2 \sinh^2(h/\Re)\,.\,d\theta^2 \equiv dL^2 \quad \text{(say)}, \qquad (80)$$

so that the system-velocity of light dL/dt is constant and isotropic, and the whole of optics can be treated by Lobatchevskyan trigonometry in *the representative* hyperbolic space (80), a 'map', as it were, of magnification ratio varying from point to point, in which straight lines stand for light rays. Thus, e.g., *the parallax formula* for a star placed at a distance h from the sun will be that given by Lobatchevsky a hundred years ago, viz.

$$\tan p = \sinh \alpha \cot \frac{h}{\Re}, \quad . \quad . \quad . \quad (81)$$

where $\alpha = a/\Re =$ astr. unit$/\Re$, or, since α is hardly greater than 10^{-10},

$$\tan p = \alpha \coth \frac{h}{\Re},$$

or, ultimately, returning to the original coordinate $\sigma = r/\Re$, through (78), and remembering that p is always but a small fraction,

$$p \doteqdot \tan p = \alpha \operatorname{cosec} \sigma. \quad . \quad . \quad (81\,a)$$

For small σ we have $\tan p = a/r$, the classical for-

system of reference $[t, h, \theta]$ is the space with constant negative curvature or hyperbolical space'. For details see *Theory of Relativity*, p. 524.

mula, while for more distant objects (81 a) differs from this as well as from Einstein's parallax formula, which has $\cot\sigma$ instead of $\operatorname{cosec}\sigma$. Accordingly, while Einstein's (cylindrical-world) formula sets to the parallax no lower limit differing from zero, de Sitter's formula leads to a *minimum parallax* $\tan p = \alpha$ or, to all practical purposes,

$$p_{min.} = \alpha = a/\Re. \quad . \quad . \quad . \quad (81\,b)$$

This would be attained by a star placed at the polar of O. Of course, even apart from the circumstance that light from such a star would require an infinite t-interval to reach us, de Sitter's minimum parallax, which cannot exceed much $10^{-12} = 0''{\cdot}0000002$, cannot be expected to lead to any observable effects. This is also true of the theoretical difference between (81 a) and the classical formula for any distances

$$r < \tfrac{1}{2}\pi\Re.$$

Before leaving these fundamental questions concerning the empty isotropic world, let us still return to the particular case of free radial motion, corresponding to a negligible β_0 and having its acceleration expressed by formula (74_0). If $\sigma = r/\Re$ does not exceed a few angular degrees, that formula gives the acceleration (whether for an approaching or a receding particle)

$$\frac{d^2r}{dt^2} = \omega^2 r, \quad \omega = c/\Re, \quad . \quad . \quad . \quad (82)$$

as if the particle were driven away by a centrifugal

force associated with a *spin* about the observer of angular velocity ω or of the revolution period

$$T = \frac{2\pi\Re}{c} = \frac{\text{double length of straight}}{\text{normal light velocity}}, \tag{83}$$

which, as it is at any rate a universal constant (and may in future serve as a natural time unit), I have some time ago proposed to call a *cosmic day*. Thus far, of course, the radius $\Re$ is unknown; but it may not be uninteresting to mention that if it were of the order of 10^{12} a.u., the 'cosmic day' would amount to a hundred million years. As to the spin-analogy itself, it is, of course, not quite complete, inasmuch as the formula (82) holds for *all* possible orientations of the radius vector drawn from the observer to the star under consideration. In other words, there is in our case *no axis of rotation*, but only a rotation centre, the observer's station. Now, although such a 'rotation' is not possible in three dimensions, it is in a four-dimensional space one of two well-known possibilities.

In fact, as has been known these fifty years or so, a four-dimensional homaloidal space E_4 can be 'rotated', i.e. transformed without changing the mutual distance of any two points, in two ways, viz. so that first a whole *plane* (E_2), and second, a single *point* (E_0) remains unmoved. Briefly, an E_4 can be rotated either about a fixed plane or about a fixed point.* The former of these rotations has been

* While, unlike an E_3, there is no rotation of E_4 about an axis (E_1).

utilized by Minkowski for an interesting representation of the special-relativistic Lorentz transformation and is, no doubt, known in this connexion to most of my readers. But it is the latter, i.e. a rotation about a point, which can help us in the present case.

Let us, then, contemplate for the moment a Euclidean four-dimensional space E_4 referred to x_1, x_2, x_3, x_4 as a Cartesian coordinate system, so that the squared line-element of E_4 will be

$$dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$

and $D = (x_1^2+x_2^2+x_3^2+x_4^2)^{\frac{1}{2}}$ the distance of an E_4-point from the origin. Next, introduce polar coordinates D, ϕ, θ, ψ through

$$\left.\begin{aligned} x_1 &= D\sin\psi\,.\sin\phi\sin\theta \\ x_2 &= D\sin\psi\,.\sin\phi\cos\theta \\ x_3 &= D\sin\psi\,.\cos\phi \\ x_4 &= D\cos\psi \end{aligned}\right\} \quad . \quad . \quad (84)$$

(giving $D^2 = \Sigma x_i^2$ identically), and apply to this fourfold the transformation

$$\psi = \psi' + \omega t'. \quad . \quad . \quad . \quad . \quad (85)$$

Since this leaves D, ϕ, θ intact, it will manifestly represent a four-dimensional rotation leaving only the origin O fixed, in brief, *a rotation about the point O*, as required for the model in hand. Our three-space (as section of the isotropic world) is elliptic, i.e. a hypersphere S_3, of radius $\mathfrak{R}$, in E_4. Yet, since we have already limited ourselves to *small* values of $\sigma = r/\mathfrak{R}$ or to a small portion of S_3, we may consider it, approximately, as a plane (E_3)

in E_4. Without any loss to generality this 'plane' (i.e. our three-dimensional space region) can be taken to be the E_4-plane $\psi' = \text{const.} = \frac{1}{2}\pi$. Now, the Galileian (homaloidal) *five*-dimensional line-element

$$ds^2 = c^2\,dt^2 - (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$$

or, by (84),

$$ds^2 = c^2\,dt^2 - dD^2 - D^2\{d\psi^2 + \sin^2\psi\,(d\phi^2 + \sin^2\phi\,d\theta^2)\},$$

is transformed by (85), with $t' = t$ and omitting all other dashes, into a quadratic form which, for the sub-manifold $\psi = \frac{1}{2}\pi$, becomes (in view of

$$D_{\frac{\pi}{2}}^2 = x_1^2 + x_2^2 + x_3^2 = r^2)$$

$$ds^2 = c^2 dt^2\left(1 - \frac{\omega^2 r^2}{c^2}\right) - dr^2 - r^2\cos^2\omega t\,(d\phi^2 + \sin^2\phi\,d\theta^2), \quad (86)$$

or; omitting the square and higher powers of

$$\omega t = ct/\Re,$$

$$ds^2 = c^2 dt^2\left(1 - \frac{\omega^2 r^2}{c^2}\right) - dr^2 - r^2\,(d\phi^2 + \sin^2\phi\,d\theta^2). \quad (86')$$

But since $g_4 = \cos^2\sigma = 1 - \sigma^2 + \ldots = 1 - r^2/\Re^2 + \ldots$, and $\Re^2\sin^2\sigma = r^2(1 - \frac{1}{3}\sigma^2 + \ldots)$, this coincides, up to σ^4 in the first and up to σ^2 in the last term, with de Sitter's line-element for distances amounting to but a small fraction of the curvature radius, provided that

$$\omega = c/\Re.$$

We thus see that, under the limitations expressly stated, the centrifugal tendency manifesting itself,

at not excessively large distances ($\sigma \ll \pi/4$), in the isotropic spacetime can be imitated by a homaloidal three-dimensional platform spinning uniformly, and with majestic slowness, in a homaloidal four-space about the observer, the period of revolution being a 'cosmic day' or twice the total length of an elliptic straight divided by the standard light velocity. Although one cannot fail to feel allured by the aesthetics and grandeur of this picture (and, as a matter of fact, the writer has often surprised himself in reacting to it, emotionally, as a reality), one must not forget that it is but a rough, though not uninstructive, model, not to be exaggerated in its significance.

Having now familiarized ourselves with the fundamental properties of the spacetime under consideration, almost—I dare say—to the extent of experiencing of it a distinct cosmic emotion,* we will pass, in the next Part, to develop certain consequences of these properties which promise to be accessible to a definite astronomical test and to lead, at the same time, to an estimate of the curvature invariant (radius) of that spacetime.

* A term due to William Kingdon Clifford. See Clifford's essay of 1877 on 'Cosmic Emotion', in his inestimable *Lectures and Essays*, Macmillan & Co., London.

PART V

DOPPLER-EFFECT IN ISOTROPIC SPACE-TIME. ATTEMPTED DETERMINATION OF ITS CURVATURE RADIUS. GRAVITATION, AND A STABILITY CRITERION OF GALAXIES

LET us consider a light source, say a star or rather one of its light-emitting atoms. Let L' be the worldline of this source, and L the worldline of the observer who will be supposed to carry with him a similar atom and to be able to compare its spectrum with the stellar one taken, of course, from his platform. Let the source emit a pair of light signals separated by the interval ds' of its proper time, say while the source passes through the points L_1' and L_2' of its worldline. These signals will be received by the observer at some points, $L_1(s)$ and $L_2(s+ds)$ say, of his worldline. In other words, let L_1, L_2 be the crosses of L with the light-cones (in Minkowski's phraseology, 'fore-cones') with L_1' and L_2' as apices. Now, our observer, receiving these messages separated by the interval ds of his proper time, will compare it with the period, say ds_0, of his local sample of the atom, and will, accordingly, establish a shift of some definite spectrum line determined by

$$\frac{\lambda+\delta\lambda}{\lambda}=\frac{ds}{ds_0}.$$

Clearly, he cannot measure the sending interval ds'

as such. But if he accepts, after Einstein, the principle of permanence of atoms, claiming

$$ds' = ds_0,$$

his spectrum-shift formula will become

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{ds}{ds'}$$

or

$$\frac{\delta\lambda}{\lambda} = \frac{ds}{ds'} - 1 = (\overline{L_1 L_2} : \overline{L_1' L_2'}) - 1. \quad . \quad (87)$$

This general expression for what may be called the (total) *Doppler effect,** which was first suggested by H. Weyl, has manifestly the precious property of being invariant with respect to any transformations of the four coordinates.

Let us now apply this general formula to the case in which both the worldlines L and L' are geodesics of the empty spacetime, that is to say, both the observer's station and the source move as free particles in absence of gravitation, or neglecting the effect of gravitation. E.g. let the former be the sun's centre and the source any star, intra- or extragalactic. We may place the origin O of the coordinates in the star. Then the worldline L' will be simply $r = 0$, and can be represented in a graph, such as that given in Fig. 8,† by the rectilinear t-axis or, more con-

* Including, e.g., without discrimination, what is due to distance and what to the relative motion of source to observer.

† This figure, borrowed from my book on *Relativity,* is drawn correctly for a radially receding star and for $k = 1$, with $ct/\Re$ as abscissae and $\sigma = r/\Re$ as ordinates.

veniently, $ct/\mathfrak{R}$-axis. The observer will describe some other geodesic, that is to say, any worldline L given for the most general case by the equations (65) and (66). Let r and $r+dr$ be the radial coordinates of the worldpoints L_1 and L_2 (representing the arrival of the aforesaid two flashes of light). Since at these points

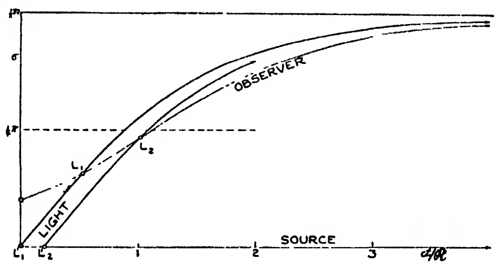

FIG. 8.

the observer is reached by the spherical wave centred at the star O when its radius is r and $r+dr$ respectively, we have

$$cdt - ds' = \sec\sigma \,.\, dr,$$

where dt is the t-separation of L_1 and L_2, so that dr/cdt is equal to the right-hand member of (66). Thus,

$$\frac{cdt}{ds'} = \frac{1}{1-\frac{1}{c}\frac{dr}{dt}\sec\sigma}.$$

But, by the significance of the original integration constant k, $cdt = kd\dot{s}/\cos^2\sigma$, which enables us to

eliminate at once dt from the last equation. Thus, and by (87), we find the required Doppler effect,

$$\frac{\delta\lambda}{\lambda} = \frac{\cos^2\sigma/k}{1 - \frac{1}{c}\frac{dr}{dt}\sec\sigma} - 1, \quad . \quad . \quad . \quad (88)$$

where the radial component of the velocity of the observer relatively to the star, or also vice versa, dr/dt, is a known function of r or σ, given by equation (66). Notice that r and dr/dt appearing in our Doppler-effect formula both refer to *the moment of receiving* the star's light. The developed form of (88), giving the effect as function of r and the integration constants k, p of the observer's worldline, is

$$1 + \frac{\delta\lambda}{\lambda} = \frac{\cos^2\sigma/k}{1 \mp \sqrt{1 - \frac{\cos^2\sigma}{k^2} - \frac{p^2\cot^2\sigma}{\Re^2 k^2}}}, \quad . \quad (88')$$

the upper sign corresponding to a receding, and the lower to an approaching motion.

In much the same way, taking $r = 0$ as the observer's worldline, or placing the origin of coordinates at *his* station, and considering the worldline of the star as any geodesic determined by (65), (66), one would find for the Doppler effect

$$1 + \frac{\delta\lambda}{\lambda} = \frac{k}{\cos^2\sigma}\left[1 + \frac{dr}{c\,dt}\sec\sigma\right], \quad . \ (88\,a)$$

and, by (66),

$$1 + \frac{\delta\lambda}{\lambda} = \frac{k}{\cos^2\sigma}\left[1 \pm \sqrt{1 - \frac{\cos^2\sigma}{\Re^2} - \frac{p^2\cot^2\sigma}{\Re^2 k^2}}\right., (88\,a')$$

where, however, r, dr/dt are to be taken for the instant of *emission* of the light,* instead of its arrival. In most actual cases this distinction (although implying a difference of tens or hundreds of thousands of years) will, of course, be only of a purely academic interest.

The values of the constants k, p characterizing the relative free motion of star and observer may conveniently be expressed in terms of σ_0 and β_0 fixing the perihelion distance of the star and its speed in passing through it. The corresponding formulae, covering the most general case, follow at once from the original definitions of k, p and from

$$(ds/cdt)_0^2 = \cos^2\sigma_0 - \beta_0^2.$$

They are

$$k = \frac{\cos^2\sigma_0}{\sqrt{\cos^2\sigma_0 - \beta_0^2}}, \qquad p = \frac{\Re\beta_0 \sin\sigma_0}{\sqrt{\cos^2\sigma_0 - \beta_0^2}}. \tag{89}$$

For small $\sigma = r/\Re$ and, *a fortiori*, small σ_0, we have $k = (1-\beta_0^2)^{-\frac{1}{2}}$, and (88) becomes, up to σ^2-terms,

$$1 + \frac{\delta\lambda}{\lambda} = \frac{\sqrt{1-\beta_0^2}}{1 - dr/cdt},$$

and this reduces, by (88′), to

$$1 + \frac{\delta\lambda}{\lambda} = \frac{\sqrt{1-\beta_0^2}}{1 \mp \beta_0},$$

the upper and the lower signs corresponding to a

* Formulae (88) and (88′) were first derived, by a somewhat roundabout method, in *Phil. Mag.*, vol. 48, 1924, p. 619, and the last two, equivalent ones, in a rejoinder to Chazy published in *Phil. Mag.*, vol. 3, 1927, p. 1085.

receding and an approaching motion respectively. In the former case, e.g., we can write

$$1 + \frac{\delta\lambda}{\lambda} = \sqrt{\frac{1+\beta_0}{1-\beta_0}},$$

which, for an orbit passing through O, agrees with the special relativistic Doppler formula for a purely radial motion, since $\beta = \frac{1}{c}\frac{dr}{dt}$ differs from β_0 only by terms of the second and higher orders of σ.

To have another special example assume, in ($88a$), $\beta_0 = 0$ and $dr/dt = 0$, i.e. a star in radial motion just reaching its perihelion (at the moment of *emission*, that is). Then that formula will reduce to

$$\frac{\delta\lambda}{\lambda} = \frac{1}{\cos\sigma} - 1 \doteq \tfrac{1}{2}\frac{r^2}{\mathfrak{R}^2},$$

agreeing with de Sitter's effect (64). Such, in fact, should be the case. For the star is, at the time of light emission, at rest relatively to the observer.

But the important thing in connexion with our main object is the development of the rigorous formula ($88a$), or ($88a'$), in powers of β_0 and σ_0, σ, for any—oblique or radial—orbit, pushed so far as to show the effect of a finite curvature radius. This, with the values (89) of the orbit constants k, p, gives, for the Doppler effect $D = \delta\lambda/\lambda$, after some simple reductions, up to higher terms, the comparatively simple formula

$$D^2 = \left(1 - \frac{r_0^2}{r^2}\right)\left(\frac{v_0^2}{c^2} + \frac{r^2}{\mathfrak{R}^2}\right), \quad . \quad . \quad . \qquad (90)$$

where, as before, r_0 is the distance of the star's perihelion, and v_0 the speed of its passage through that particular point of its orbit, while r is, apart from niceties, the actual distance of the star from the observer.* This formula holds for *any* free orbit, straight and therefore radial, or oblique and accordingly resembling a hyperbola (*v. supra*). For a *receding* star D, the square root of (90), is, in accordance with (88′), to be taken with the *positive*, and for an approaching one, with the *negative* sign. Notice, also, that there is nothing inherent in the isotropic spacetime to favour the frequency of occurrence in Nature of the former as against that of the latter case—any assertions to that effect (as e.g. those due to H. Weyl) being of a purely speculative kind, lacking every serious support by either 'first principles' or astronomical observations.

Before deriving conclusions from this general equation, which will occupy much of our attention in the remainder of this book, let us discuss its implications in a few simple cases.

Firstly, if the orbit of the star passes actually through the origin O,† so that it is a straight line of the elliptic space, and the star's velocity is through-

* More accurately its distance at some moment between the emission and the arrival of the light under examination.

† Station of a fictitious observer, practically (after the usual reduction of spectroscopic radial velocity $V = cD$ to the sun) $O =$ centre of sun.

out radial (Fig. 9*a*), we have $r_0 = 0$, $v_0 \neq 0$, and (90) becomes

$$D \equiv \frac{V}{c} = \pm\sqrt{\frac{v_0^2}{c^2} + \frac{r^2}{\Re^2}}. \quad . \quad . \quad (90_1)$$

The 'effect' is thus seen to be of the *first order*, not only in v_0/c but also in $r/\Re$. The correlation of signs is as above.

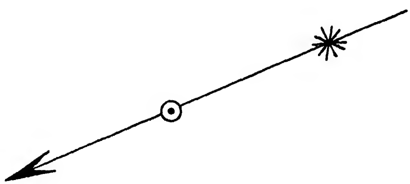

FIG. 9 *a*.

Secondly, if the orbit is radial, and thus also rectilinear, but ends at a finite perihelion, $r_0 \neq 0$ (Fig. 9 *b*), we have $v_0 = 0$ (turning-point) and

$$D^2 = \frac{r^2 - r_0^2}{\Re^2}. \quad . \quad . \quad . \quad (90_2)$$

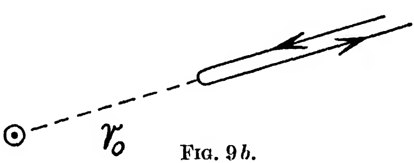

FIG. 9 *b*.

In particular, at a time when r is much greater than r_0,

$$D = \pm\frac{r}{\Re},$$

and the spectroscopic radial velocity $V = \pm cr/\Re$.

Lastly, if the orbit is of the general type (resembling a hyperbola) and does not lead through the origin, as in Fig. 9 *c*, but if we confine our attention

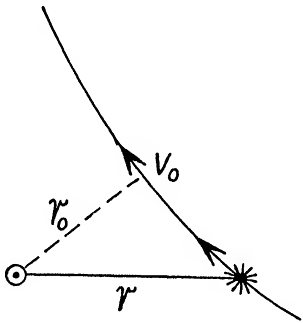

FIG. 9 *c*.

to small $r/\mathfrak{R}$, which presupposes small $r_0/\mathfrak{R}$, then σ^2 may be neglected in presence of $\beta_0^2 = v_0^2/c^2$, and we are left with

$$V = v_0\sqrt{1 - r_0^2/r^2}, \quad . \quad . \quad . \quad (90_3)$$

which is simply the classical result, the contemplated branch of the orbit being confounded with an elliptic straight line and this with a Euclidean one. In fact, classically, $v = \text{const.} = v_0$ (uniform motion), so that V as given by (90_3) is simply the radial component of the resultant orbital velocity of the star.

After these illustrative examples we may return to the general formula (90), itself only an approximation to the rigorous one (88′), but one which will be accurate enough for all astronomical applications.

With V_r or simply $v_r = dr/dt$ (from which it is

practically indistinguishable) written for the *spectroscopic radial velocity*, supposed to be reduced to the sun in the usual way, our formula becomes

$$v_r^2 = \left(1 - \frac{r_0^2}{r^2}\right)(v_0^2 + c^2\sigma^2), \qquad . \quad . \quad (90)$$

where $\sigma = r/\Re$. This will in the sequel be our main working formula.

In certain cases, though only very rare ones, the transversal velocity of remote celestial objects* (stars or nebulae) is known from observation, if, that is, the angular velocity or the 'proper motion' (μ) and the distance of the object have been measured. It is therefore not without interest to note here the formula for *the transversal angular velocity* $d\theta/dt = \mu$. This is, rigorously, by (65) and (89)

$$\mu = \frac{v_0}{\Re} \frac{\cos^2\sigma}{\cos^2\sigma_0} \frac{\sin\sigma_0}{\sin^2\sigma},$$

whence, the (linear) transversal velocity,

$$v_t = \mu\Re\sin\sigma,$$

$$v_t = v_0 \cdot \frac{\cos^2\sigma}{\cos^2\sigma_0} \cdot \frac{\sin\sigma_0}{\sin\sigma}. \qquad . \quad . \quad . \quad . \quad (91)$$

For small σ the last factor can be written $\sigma_0/\sigma = r_0/r$, and the segment of the orbit from the perihelion to the actual position of the star, say $\Re\lambda$ (Fig. 10), can be considered as that of a straight line of the elliptic space, so that, by a fundamental

* Manifestly the nearer stars, placed say at ten parsecs from our station, are of no interest in the present connexion.

theorem of elliptic trigonometry, $\cos^2\sigma/\cos^2\sigma_0 = \cos^2\lambda$, and (91) becomes

$$v_t = \frac{v_0 r_0}{r}\cos^2\lambda. \qquad (91\,a)$$

Turning at length to astronomical applications of our formulae, let us first notice that the two star-constants, r_0 and v_0, appearing in (90), and especially the former, are, and possibly will remain yet a long time, unknown for any individual star. For to state that a certain distance is the minimum distance of a given and rather remote star from our sun, one would have to observe its position for many thousand years. And not knowing the star's r_0, one cannot assert anything about its v_0, the velocity of its passage through the perihelion. In fine, then, the individual star-constants v_0, r_0 are to be considered as unknown and, practically, unknowable. If the proper motion μ is observed, as well as the actual distance r, then at least a numerical relation between v_0 and r_0 will be available, leaving but one of these two constants open to conjecture. But even this will (for the more distant objects) happen only in sporadic cases. Thus (91) will not help us much in determining the 'Size of the Universe', i.e.

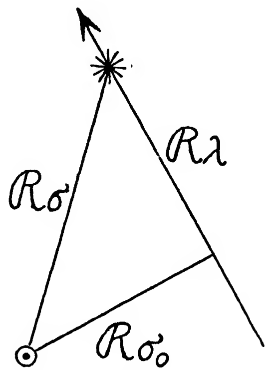

FIG. 10.

the radius $\mathfrak{R}$. And the same, of course, is true of the formula for the resultant velocity $v^2 = \sqrt{v_r^2 + v_t^2}$, which can readily be shown to be

$$v^2 = c^2 \cdot \cos^2 \sigma \{1 - (1 - \beta_0^2) \cos^2 \lambda\} \quad . \quad (92)$$

Consequently our attention will, in general, have to be fastened upon the Doppler-effect formula (90), with its two inexplorable *individual* star-constants v_0, r_0.

But, in spite of this difficulty, a good deal of information and even a probable estimate of the value of the *universal* constant $\mathfrak{R}$ can be derived from this formula, namely, by applying the well-known methods of statistics. The reader will remember that modern astronomy, especially *stellar* astronomy, abounds in such statistical procedures, and that, accordingly, most of its results are, in fact, of a statistical nature.

Imagine, then, that a certain, not too small, number (n) of distant celestial objects, of known distance r and radial velocity v_r, has been picked out *at random*. Then it will be reasonable to assume, first of all, that the v_0 are distributed among them at random. Similarly, in the next place, will the values of r_0 or, let us better say, those of the equally unknown ratio r_0/r, have a random distribution among the n objects, in this case, of course, all being contained within the limits

$$0 \leqslant \frac{r_0}{r} \leqslant 1,$$

inasmuch as r_0 can have any value from 0 to $\frac{1}{2}\pi\mathfrak{R}$, and r from r_0 to $\frac{1}{2}\pi\mathfrak{R}$.

Now, as nothing at all is known about r_0/r (apart from these limits themselves), it is reasonable to assume that *all* values of $x = r_0/r$ from 0 to 1 are *equally probable*, regardless of the value of r itself. Thus the number of stars whose x-values range from x to $x+dx$ will be ndx, and the average value of $x^2 = (r_0/r)^2$, which occurs in (90),

$$\int_0 x^2 dx = \frac{1}{3},$$

and therefore the average or mean of the right-hand member of equation (90) would be

$$\tfrac{2}{3}\,(v_0^2 + c^2\sigma^2),$$

under the assumption, of course, that all n objects have the same v_0. Such, however, will not be the case. On the contrary, from what we know about near-by stars (whose actual velocities cannot differ much from their v_0's), these values will be different for the n members of the contemplated group. But here again it is reasonable to assume that the v_0-values have a Maxwellian distribution among the stars,* i.e. that the number of stars whose perihelion velocities fall within the range dv_0 is

$$An\, e^{-\kappa v_0^2}\, dv_0,$$

* Which amounts to supposing that the group is in statistical equilibrium.

where A and κ are constants, of which the former is determined by

$$A\int_0^\infty e^{-\kappa v_0^2}\,dv_0{}^* = \frac{\sqrt{\pi}}{2\sqrt{\kappa}}A = 1,$$

so that the last-said number becomes

$$2n\sqrt{\frac{\kappa}{\pi}}\,e^{-\kappa v_0^2}\,dv_0. \quad . \quad . \quad . \quad (93)$$

Thus, the average of v_0^2 will be

$$\overline{v_0^2} = 2\sqrt{\frac{\kappa}{\pi}}\int_0^\infty v_0^2\,e^{-\kappa v_0^2}\,dv_0 = \frac{1}{2\kappa}, \quad . \quad (93\,a)$$

and, averaging both sides of (90),

$$\overline{V_r^2} = \frac{2}{3}\left(\overline{v_0^2} + \frac{c^2}{\Re^2}\overline{r^2}\right),$$

where $\overline{r^2}$ as well as $\overline{V_r^2}$ are supposed to be known from observation, while c is the familiar universal constant, 3.10^{10} cm./sec. The only unknown, barring our way to the much-desired $\Re$, is the mean $\overline{v_0^2}$. This is, under (93 a), expressed in terms of the coefficient of precision, as κ is called. But this in its turn is not directly known. Now the (arithmetical) mean of v_0 itself is, by the distribution-law (93),

$$\overline{v_0} = 2\sqrt{\frac{\kappa}{\pi}}\int_0^\infty v_0\,e^{-\kappa v_0^2}\,dv_0 = \frac{1}{\sqrt{\pi\kappa}},$$

so that, by (93 a),

$$\overline{v_0^2} = \frac{\pi}{2}(\overline{v_0})^2.$$

* The upper limit of v_0 cannot, of course, exceed some finite value, but if this is only a few times the mean value, the result will be practically the same as for an infinite upper limit of the integral.

Ultimately, therefore, our formula gives the statistical result

$$c^2\overline{D^2} \equiv \overline{V_r^2} = \frac{2}{3}\left\{\frac{\pi}{2}(\overline{v_0})^2 + \frac{c^2}{\Re^2}\overline{r^2}\right\}, \quad . \quad (94)$$

where $\overline{v_0}$ is the mean of the unknown individual v_0-values.

Now, in absence of any knowledge of these values, the only thing one can do is to assume that $\overline{v_0}$ is the same as *the mean 'space-velocity'* (i.e. resultant velocity) of near-by stars, that is, in round figures,

$$\overline{v_0} = 30 \text{ km./sec.}$$

This would give, ultimately, for the calculation of the curvature radius $\Re$ the formula

$$\frac{1}{\Re^2} = \frac{3\overline{D^2} - \pi . 10^{-8}}{2\overline{r^2}}.$$

In the second term of the numerator π can as well be replaced by 3, especially as but little accuracy can be attributed to the adopted value of $\overline{v_0}$. Thus the formula becomes

$$\Re^2 = \frac{2}{3} \cdot \frac{\overline{r^2}}{\overline{D^2} - 1.10^{-8}} \quad . \quad . \quad . \quad (94\,a)$$

Notice that the subtractive term 10^{-8} is by no means a superfluous nicety. For if the observed radial velocities are even of the order of 100 km./sec., $\overline{D^2}$ will be of the order 10^{-7}, and for 30 km./sec. of the

order 10^{-8}. In fine, the two terms are comparable in importance.

We will now, before proceeding with some further generalities, apply the last formula to the data concerning eighteen globular clusters and the two Megellanic Clouds which constituted the total material at my disposal in August 1924,* and which —in spite of strenuous efforts to induce the Mount Wilson and other Observatories to take the spectrograms of some further globular clusters of known distance—I was up to the present (December 1928) unable to amplify. These data are collected in Table I, of which the first column gives the N.G.C. numbers of the globular clusters and the abbreviated denominations of the two Magellanic Clouds (L = Lesser, G = Greater); the second column the distance in kiloparsecs, as measured by Shapley; and the third the D-effects based almost all on M. V. Slipher's spectrographic results. The probable error of most of these D-data, relating to the clusters, is, according to Prof. Slipher's private letter (of 28 July 1924), not less than ± 10 km./sec., and in the case of N.G.C. 6934 and 6229, marked (?), very likely much greater. The total of twenty objects is ordered

* Toronto Meeting of the British Association and the International Congress of Mathematicians, before which this subject was brought up, 13 and 17 August. The nebula N.G.C. 598 (Messier 33), until then placed by Hubble at 31·3 kiloparsecs, has about New Year 1925 been declared by him to be about *nine* times as distant, and is now omitted in this connexion.

according to ascending distance. The bracketed numbers (1), (2), &c., are just affixed for the sake of brief reference.

TABLE I

Object.	r	D . 10^4	Object.	r	D . 10^4
(1) 6205	11·1	−10·0	(11) 1851	17·2	+10·0
(2) 6341	12·3	−5·3	(12) 6626	18·5	zero
(3) 6218	12·4	+5·3	(13) 5024	18·9	−5·7
(4) 5904	12·5	$+0{\cdot}3_3$	(14) 6093	20·0	+2·3
(5) 5272	13·9	−4·2	(15) 6333	25·0	+7·5
(6) 7078	14·7	−3·2	(16) 1904	25·6	+6·7
(7) 6266	15·2	−1·7	(17) L.M.Cl.	31·0	+5·6
(8) 7089	15·6	$-0{\cdot}3_3$	(18) 6934	33·3	−11·7 (?)
(9) 6273	15·9	+1·0	(19) G.M.Cl.	35·0	+9·2
(10) 7099	17·2	−4·1	(20) 6229	43·5	−3·3 (?)

We may note, for the sequel, that the mean distance and mean absolute D-effect for these twenty objects are

$$\bar{r} = 20{\cdot}5, \quad \overline{|D|} = 4{\cdot}87.$$

The mean squares needed for the application of formula (94 a) are, for all the twenty objects,

$$\overline{r^2} = 493{\cdot}0, \quad \overline{D^2} = 35{\cdot}4 \,.\, 10^{-8},$$

so that the formula gives the curvature radius

$$\mathfrak{R} = 3{\cdot}09 \,.\, 10^7 \text{ parsecs} = 6{\cdot}37 \,.\, 10^{12} \text{ astr. units.} \qquad (95_1)$$

It is interesting to notice that in this case the v_0^2-term to be subtracted from $\overline{D^2} = 35{\cdot}4 \,.\, 10^{-8}$ is only about 1/35 of the latter, so that this perihelion-velocity term may be omitted in the averaged equation (94). If we omitted it, tentatively, also in

the original formula (90), we would have, for each of the twenty objects,

$$D = \sigma\sqrt{1 - r_0^2/r^2},$$

whence

$$\left(\frac{r_0}{r}\right)^2 \doteqdot 1 - D^2/\sigma^2, \quad . \quad . \quad . \quad . \quad (96)$$

and since $\mathfrak{R}$, and therefore σ, are now known, the ratio

$$x = r_0/r$$

can be calculated for each object. This seems worth while, since, first, the factor $(1 - x^2)$ influences the effect much stronger than do the oscillations of v_0^2 around a mean value, and second, the value of $\mathfrak{R}$ obtained, no matter how likely on other grounds, should with (96) give for every object a *real* x or positive r_0^2/r^2, and this is obviously worth testing. Now, using (95_1), for $\mathfrak{R}$, one finds, by (96), for the objects (1), (2) . . . (20) respectively,

$$\left(\frac{r_0}{r}\right)^2 = \mathit{-8{\cdot}54} \quad \mathit{-0{\cdot}77} \quad \mathit{-0{\cdot}74} \quad 0{\cdot}994 \quad 0{\cdot}16 \quad 0{\cdot}55 \quad 0{\cdot}88 \quad 0{\cdot}996$$
$$0{\cdot}96 \quad 0{\cdot}46 \quad \mathit{-2{\cdot}22} \quad 1{\cdot}00 \quad 0{\cdot}14 \quad 0{\cdot}87 \quad 0{\cdot}17 \quad 0{\cdot}31 \quad 0{\cdot}69$$
$$\mathit{-0{\cdot}19} \quad 0{\cdot}34 \quad 0{\cdot}95.$$

Five of these values, corresponding to N.G.C. 6205, 6341, 6218, 1851, and 6934 (for the last of which the D-value is rather uncertain), would give imaginary r_0 if, as we have just done, their v_0's were neglected. The question naturally arises whether this can be remedied when their v_0's are taken into account, and if so, what v_0-values are to be attributed to these clusters to straighten things out.

Take first N.G.C. 6205, the worst of the five rebels. The smallest v_0 needed to make r_0 real at all will, of course, be, by (90), that corresponding to $r_0 = 0$. Thus we shall have

$$V_r^2 = v_0^2 + c^2\sigma^2$$

or

$$\dot{D}^2 = \beta_0^2 + \sigma^2,$$

whence

$$\beta_0^2 = 10^{-6} - 0{\cdot}129\,.\,10^{-6} = 0{\cdot}87\,.\,10^{-6},$$

i.e.

$$v_0 = 282 \text{ km./sec.}$$

That is to say, if we assume that this cluster has a purely radial motion ($r_0 = 0$), or is comparatively far away from its perihelion (r_0/r small), we have to claim that when it will in some distant future come closest to our sun, its velocity, which now has the radial component -300, will be reduced to -282 km./sec. There certainly is nothing fantastic or even unlikely about such an implied prediction.

Similarly the reader may treat the remaining four clusters. And there will again be nothing unlikely about his results.

In fine, then, the above-found size, $6{\cdot}4 \,.\, 10^{12}$ astr. units of the curvature radius is not incompatible with any of the twenty pairs of data, r, D. Notice that the vanishing effect $D = 0{\cdot}0$, for the cluster 6626, in spite of its large distance, 18·5 kiloparsecs, is naturally accounted for by $r_0 = r$.* That is to say, we infer, by (90), from its spectroscopic behaviour that it just passes through its perihelion or nearly so, or that its motion is now purely transversal. There is certainly nothing surprising in the fact that one out of twenty objects should just be at or near its perihelion. The same is, less closely, the case of 5904 and 7089, which, accordingly, are given the values $r_0^2/r^2 = 0{\cdot}994$, $0{\cdot}996$, differing but little from unity. Finally, notice that two of the objects have been allotted the values 0·31, 0·34 of r_0^2/r^2 which are very close to the mean, $\frac{1}{3}$, adopted in formula (94). And, as concerns the remaining objects, a scattering of their r_0^2/r^2 around this mean is what one would expect beforehand.

So much as to the interpretation of the list of values of that ratio derived by assuming provisionally that the v_0^2-term in formula (94) is entirely negligible (and then amending this assumption in some individual cases).

* As we know, the rigorous formula (88 a) gives, in this case of transversal motion, a certain non-vanishing effect, but this is of the second order, namely,

$$D \doteq \tfrac{1}{2}\sigma^2 = 1{\cdot}7 \,.\, 10^{-7},$$

or 0·051 km./sec., which is much too small to be ever detected.

Another plan, and perhaps a better one, for satisfying the conditions imposed by all the observed data seems to be the following one.

Return to formula (94), which can be re-written

$$\overline{D^2} = \frac{2}{3}\left(\overline{\beta_0^2} + \frac{1}{\Re^2}\overline{r^2}\right), \quad . \quad . \quad . \quad (97)$$

and apply it to two sub-groups of the whole group of objects, one on the whole more, and the other less distant, to be distinguished by the suffixes 1 and 2. Then

$$\overline{D_1^2} = \frac{2}{3}\left(\overline{\beta_{01}^2} + \frac{1}{\Re^2}\overline{r_1^2}\right)$$

$$\overline{D_2^2} = \frac{2}{3}\left(\overline{\beta_{02}^2} + \frac{1}{\Re^2}\overline{r_2^2}\right).$$

Subtract these from each other assuming that $\overline{\beta_{01}^2} = \overline{\beta_{02}^2}$, which is reasonable, since β_0 does not depend on the actual distances of the objects. Thus,

$$\Re^2 = \frac{2\,(\overline{r_1^2} - \overline{r_2^2})}{3\,(\overline{D_1^2} - \overline{D_2^2})}, \quad . \quad . \quad . \quad (98)$$

a formula for determining $\Re$ without invoking any data extraneous to the group itself.

Let us apply this to two groups, each consisting of *ten* objects, viz. (1) to (10), (11) to (20), of Table 1. Then

$$\left.\begin{aligned}\overline{r_1^2} &= 202{\cdot}3\\ \overline{r_2^2} &= 783{\cdot}8\end{aligned}\right\}\ (\text{kilopars.})^2, \qquad \begin{aligned}\overline{D_1^2} &= 20{\cdot}5\,.\,10^{-8}\\ \overline{D_2^2} &= 50{\cdot}3\,.\,10^{-8}\end{aligned}$$

and

$$\left.\begin{aligned}\Re &= 3{\cdot}61\,.\,10^{7}\ \text{parsecs}\\ &= 7{\cdot}44\,.\,10^{12}\,\text{astr. units}\end{aligned}\right\}, \quad . \quad . \quad . \quad (98_1)$$

by about 17 per cent. only greater than the last-obtained one.

If this radius be introduced into either sub-group, we have for the mean-square velocity $\overline{v_0^2} = c^2\overline{\beta_0^2}$ at the perihelia, which should be common to both,

$$\overline{\beta_0^2} = \tfrac{3}{2}\overline{D_1^2} - \overline{\sigma_1^2} = \tfrac{3}{2}\overline{D_2^2} - \overline{\sigma_2^2} = 15{\cdot}2\,.\,10^{-8}\,,$$

i. e.

$$\sqrt{\overline{v_0^2}} = 117\ \text{km./sec.}$$

If we attribute this value of $\overline{v_0^2}$, that is

$$v_0 = \overline{v_0} = 93{\cdot}6\ \text{km./sec.}$$

(by no means an excessive velocity), to each object of the whole group, we have, by (90),

$$D^2 = (1-x^2)\,(\sigma^2 + 15{\cdot}2\,.\,10^{-8}), \quad . \quad . \qquad (99)$$

whence we can calculate $x = r_0/r$ for each of them. This gives the following x^2-values for the twenty objects of Table I. In the last column *the least* v_0-values (in km./sec.) are given for those objects to which the value $v_0 = 93{\cdot}6$ cannot be attributed. Of course, even the remaining objects need not have each $v_0 = 93{\cdot}6$ km./sec., and their v_0's are, no doubt, scattered around that value. But we see from this table that the radius (98_1), with the reasonable value $\sqrt{\overline{v_0^2}} = 117$, is quite compatible with all the individual pairs of observed radial velocities and distances.

We have spent so much time upon this and the preceding set of numerical results in order to bring home to the reader *the very considerable influence of*

the unknown factor $1-r_0^2/r^2$ in our expression for the Doppler effect of an individual celestial object. This factor, oscillating between 0 and 1, together with

TABLE II. ($\mathfrak{R} = 3{\cdot}61 \,.\, 10^7$ parsecs)

Object.	$x^2 = r_0^2/r^2$	$x = 0$ Least v_0	Object.	$x^2 = r_0^2/r^2$	$x = 0$ Least v_0
(1)	(− 3·05)	285	(11)	(− 1·64)	264
(2)	(−0·05)	122	(12)	1·00	—
(3)	(−0·04)	121	(13)	0·237	—
(4)	0·996	—	(14)	0·885	—
(5)	0·412	—	(15)	0·110	—
(6)	0·682	—	(16)	0·315	—
(7)	0·912	—	(17)	0·648	—
(8)	0·997	—	(18)	(−0·37)	216
(9)	0·972	—	(19)	0·225	—
(10)	0·556	—	(20)	0·932	—

the smaller oscillations of that expression due to the other unknown star-constant v_0, can be expected to *mask very strongly* the correlation between observed radial velocity and distance predicted by equation (90), that is to say, to *lower the correlation coefficient* between these two attributes of stars or stellar systems; in other words, to produce a scattering of the $r, |D|$ points around a curve representing graphically that equation with some unique compromise values of r_0^2/r^2 and v_0^2. Only if the number of objects would be very large, say running into hundreds or possibly thousands, could one expect, under these circumstances, a huge correlation coefficient, large, that is, when compared with its own probable error.

After this warning, which seems the more necessary, as several leading astronomers, criticizing (since 1924) my attempts of thus establishing the isotropic world and estimating its size ($\mathfrak{R}$), have not taken due account of these powerful scattering causes, we may pass on to construct, on the well-known statistical principles, the correlation coefficient between r and $|D|$, the absolute value of the Doppler effect, and a curve representing summarily their quantitative interdependence.

To begin with the latter, let us assume for r_0^2/r^2 its mean value

$$x = \tfrac{1}{3},$$

and for β_0^2 the value just found and used in equation (99),

$$\beta_0^2 = 15{\cdot}2 \,.\, 10^{-8}.$$

Then that equation will become

$$V_r^2/c^2 \equiv D^2 = \frac{2}{3}\left(\frac{r^2}{\mathfrak{R}^2} + 15{\cdot}2 \,.\, 10^{-8}\right). \quad . \qquad (100)$$

The corresponding curve, with the last-obtained radius,

$$\mathfrak{R} = 36{,}100 \text{ kiloparsecs}, \quad . \quad . \quad . \qquad (98_1)$$

and r (in kiloparsecs) as abscissae and $|V_r|$ (in km./sec.) as ordinates, is drawn in Fig. 11. The centres of the circlets represent the pairs of measured spectroscopic velocities and distances as given in Table I.

The dotted circle represents the spiral N.G.C. 598 or Messier 33, with $V_r = -260$, and (up to 1924) $r = 31{\cdot}3$.

This beautiful object, fitting the curve without being consulted in its construction, has since, unfortunately, been dragged away by Dr. Hubble to something like *nine*

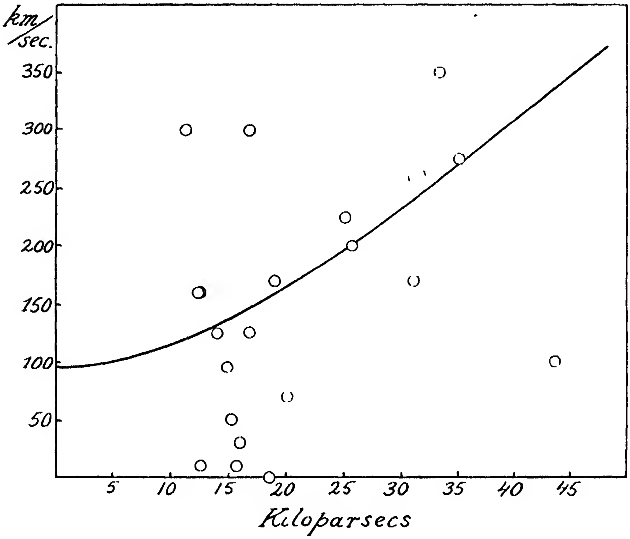

FIG. 11.

times that distance, as was first announced by Prof. Henry Norris Russell, just on New Year 1925, in a discussion on the Finiteness or Infiniteness of the Universe held at the Washington meeting of the A.A.A.S. It may be well to mention, however, that the Cordoba observers, as quoted some time ago in *Nature* by Prof. E. T. Whittaker, continue to adhere to a distance of the order of 30 kiloparsecs, or at least did so in 1927. At any rate no definite importance can be attributed to this spiral in the present connexion, although I had not the heart to remove this

true gem of the heavens from its older location. Nor has Dr. Hubble, to my knowledge, ever explained what was wrong with his original distance estimate.

Of the twenty objects, three—as passing almost through their perihelia—cannot actually be considered as giving any evidence either for or against the formula in question. They are, for the time being, inoperative. Of the remaining seventeen objects *ten** certainly lie close enough to the theoretical curve, three others are tolerably well situated, while the remaining four, N.G.C. *6229* and 6205, 1851, *6934*, are outspoken rebels, hovering much below or soaring far above that curve. Yet none of them can be said to upset the theory or, say, the isotropic cosmology. First of all the underlined ones have, according to their observer himself (Prof. Slipher), but poorly measured D's or V_r's. And, second, the negative deviation of 6229 is well explicable by an r_0^2/r^2-value considerably greater than that (1/3) adopted for the curve, in other words, by assuming it to be much nearer its perihelion, and similarly the globular clusters 6205, 1851, 6934 much farther away from their perihelia. The graph, as it stands, seems—in view of these considerations and the previous general remarks—encouraging enough (*even* with the desertion of the M. 33 spiral).

Next, as to the correlation coefficient between

* Notice that at $|V_r| = 160$ there are *two* circlets, almost overlapping ($r = 12{\cdot}3$ and $12{\cdot}4$).

$V_r|$ and r. Let ξ, η be the differences of the actual r and $|V_r|$ over the mean values $\bar{r}$, $\overline{|V_r|}$ for a group of n objects. Then the familiar Bravais-Pearson *coefficient of correlation* between the (absolute) radial velocities and distances, apart from any cosmological preconceptions, is

$$k = \frac{\Sigma \xi\eta}{\sqrt{\Sigma\xi^2 . \Sigma\eta^2}}, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the sums to be extended over the n objects, and *the probable error* of this coefficient,

$$\text{P.E.} = \frac{0{\cdot}675\,(1-k^2)}{\sqrt{n}}. \quad . \quad . \quad . \quad . \quad (b)$$

Discarding, for reasons aforesaid, the two globular clusters N.G.C. 6934, 6229, let us apply these statistical formulae to the remaining eighteen objects listed in Table I. We have, to three figures,

$$\bar{r} = 18{\cdot}4, \qquad \overline{|V_r|} = 137,$$

$$\Sigma\xi^2 = 754, \qquad \Sigma\eta^2 = 164{,}500,$$

$$\Sigma\xi\eta\,^* = 5{,}820 - 1{,}589 = +4{,}231,$$

whence, by (a), the correlation coefficient for the eighteen objects, $k = +0{\cdot}381$, and its probable error, by (b), 0·136. In fine,

$$k = +0{\cdot}381 \pm 0{\cdot}136, \quad . \quad . \quad . \quad (101)$$

which is of the right sign and nearly *three times its probable error* and can thus be considered as a fairly

* *Five* negative and *thirteen* positive items.

good indication of the existence of the predicted dependence of the spectroscopic effect on distance.* Many a correlation in stellar, *and* nebular, statistics has, especially of late, been claimed on the ground of smaller coefficients. Notice that not k itself but its ratio to its own P.E. is the true indicator of a correlation.

Of course no such universal law and the value of the implied cosmical constant $\mathfrak{R}$ can be considered as finally established on the evidence of a mere score of objects (lest we fall under the ban of the very objections we have, some pages ahead, raised against Dr. Hubble's bold extrapolation from six nebulae to myriads of 'similar' ones), and a good number of other, equally, if not more, accurate data for distant celestial objects will have to be collected and examined before settling this important question. (Has not even the determination of that much smaller quasi-constant, our mother Earth's radius, cost unspeakable toil and patience?) We hope to be able to add, in the near future, a good number more of similar pairs (r, D) of data. [See Notes at the end of the volume.] Meanwhile, the result just obtained will, I trust, encourage the reader to consider the further implications of an isotropic

* Namely, by a known theorem, the probability that this value of the correlation coefficient should be due to mere chance, is, with Φ written for the error function,

$$1-\Phi\left(0{\cdot}477\frac{k}{P.\,E.}\right) = 1-\Phi\,(1{\cdot}34) = 0{\cdot}058.$$

world characterized by a *finite* curvature radius, of the order of 10^4 kiloparsecs, implications of some general cosmological interest, to be given presently.

But even before proceeding to these, and in absence of any other material comparable to that afforded by the globular clusters (with their particularly reliable distance measurements) enabling one to form a numerical idea of the curvature radius $\Re$, I should like to give here an additional support to the claimed correlation between radial velocity and distance by utilizing the somewhat more doubtful distance relations (due to Hubble) of thirty-eight *extra-galactic nebulae* (spirals) together with their well-ascertained radial velocities tabulated some years ago by Slipher. We will take the accuracy of the (total) apparent magnitudes m_T of these nebulae, as given in Hubble's paper (discussed in Part III), for granted, also his formula (H_r), p. 95,

$$\log r = \text{const.} + 0{\cdot}2\, m_T,$$

for granted.* Yet, as the value 4·04 given by him to the additive 'constant' certainly cannot claim to be built on rock, and as we now propose to consider the $r, |D|$ *correlation* itself, which (unlike $\Re$) is manifestly *independent* of that constant, we naturally prefer to leave its value open. Thus, calling it, say, $\log \lambda$, we will write Hubble's generalized, empirical

* This amounts to assuming that all extra-galactic nebulae have the same absolute luminosity.

formula for these, and presumably other, extra-galactic nebulae,

$$\log\frac{r}{\lambda}=\frac{1}{5}m_T, \quad . \quad . \quad . \quad . \qquad (102)$$

where λ will, until further developments, be considered as an unknown distance. This seems but fair. (The 'log' stands for the common, base 10, logarithm.) To repeat it, whatever the distance λ, in miles or kiloparsecs, it will not influence the correlation coefficient k, as a glance on its definition (a) will show.

With this understanding the data for the said thirty-eight spiral nebulae are listed in Table III, the first column giving the N.G.C. labels of these objects, the second their apparent magnitudes m_T, according to Hubble's paper (*Mount Wilson*, No. 324, 1926), the third their distances with λ as (unknown) unit, and the fourth their spectroscopic velocities V_r in km./sec., as listed in V. M. Slipher's complete table of radial velocities of spiral nebulae (1922), reproduced, e.g., in Eddington's *Mathematical Theory of Relativity* (ed. 1, p. 162). The signs of V_r, though for our purposes irrelevant, are also given, since the reader may like to know them for any other reason (only five are negative). All the objects are here ordered in ascendant 'magnitude', and thus also distance. (Our old friend, N.G.C. 598, is underlined.)

The mean distance and (absol.) radial velocity are $\bar{r}/\lambda = 105$, $\overline{|V_r|} = 616$. Next, with similar meaning

TABLE III

Object	m_T	r/λ	V_r	Object	m_T	r/λ	V_r
(1) 224	5·0	10	−300	(20) 2683	9·9	96	400
(2) 598	7·0	25	−260	(21) 3623	9·9	96	800
(3) 5194	7·4	30	270	(22) 4382	10·0	100	500
(4) 3031	8·3	46	−30	(23) 3368	10·0	100	940
(5) 4736	8·4	48	290	(24) 3521	10·1	105	730
(6) 4258	8·7	55	500	(25) 4111	10·1	105	800
(7) 221	8·8	58	−300	(26) 1023	10·2	110	300
(8) 4472	8·8	58	850	(27) 5236	10·4	121	500
(9) 4826	9·0	63	150	(28) 7331	10·4	121	500
(10) 3034	9·0	63	290	(29) 584	10·9	152	1800
(11) 3627	9·1	66	650	(30) 4565	11·0	159	1100
(12) 4594	9·1	66	1100	(31) 404	11·1	166	−25
(13) 2841	9·4	76	600	(32) 4526	11·1	166	580
(14) 3379	9·4	76	780	(33) 936	11·1	166	1300
(15) 4449	9·5	$79·_5$	200	(34) 3489	11·2	174	600
(16) 3115	9·5	$79·_5$	600	(35) 4214	11·3	182	300
(17) 4649	9·5	$79·_5$	1090	(36) 5866	11·7	219	650
(18) 5055	9·6	83	900	(37) 278	12·0	251	650
(19) 4486	9·7	87	800	(38) 4151	12·0	251	980

of symbols as in the last computation, to three figures,
$\Sigma\xi^2 = 130{,}000, \quad \Sigma\eta^2 = 5{,}240{,}000,$

$$\Sigma\xi\eta = 405{,}700 - 154{,}500 = +251{,}200,$$

whence the correlation coefficient

$$k = +0{\cdot}304 \pm 0{\cdot}0994, \quad . \quad . \quad . \quad (102)$$

which, being 3·1 times its probable error, is just as good an indicator of correlation between distance and radial velocity, as was that based on the globular clusters and the Magellanic Clouds.

As was already pointed out, the additive constant in Hubble's relation, and therefore λ, being left free, this set of thirty-eight pairs of data cannot enable

us to determine the curvature radius $\mathfrak{R}$ in, say, parsecs. But it, clearly, can be utilized to determine the ratio $\mathfrak{R}/\lambda$. Splitting the whole into two subgroups, each of nineteen spiral nebulae, even as placed in the two sections of Table III, and applying formula (98), derived above, I find

$$\frac{R}{\lambda} = 6{\cdot}68 \,.\, 10^4. \quad . \quad . \quad . \qquad (103)$$

According to Hubble, as mentioned before, $\log \lambda = 4{\cdot}04$, and if one adopted this constant, one would have, by (103), $\mathfrak{R}$ = 732,000 kiloparsecs, or a little over *twenty* times our last $\mathfrak{R}$-value, (98_1). But this λ, derived ultimately from Hubble's distance estimate of N.G.C. 598, viz. 275 kiloparsecs, can scarcely claim finality. Lastly, if we apply the formula (98) to the 20 objects of Table I as Group 1, and the 38 spirals, with Hubble's λ-value, the result is $\mathfrak{R}$ = 413,000 kilopars. = $8{\cdot}52 \,.\, 10^{13}$ astr. units.

Leaving, for the present, the question of the numerical evaluation of the curvature radius, let us now turn to some of the main implications of its finiteness at all, and in the first place let us consider *Gravitation* associated with mass-points or spherical masses placed in the hitherto empty spacetime of de Sitter.

The solution of the field equations corresponding to a mass-centre has already been given in Part IV, under (62), where it has also been made plausible that the existence of such a centre does not (as in the case of Einstein's cylindrical world) give rise to 'fatal polars' or any other physical singularities.

If Lc^2 be the mass imagined condensed at the origin O of the coordinates, that solution can be re-written, in the form of the line-element of space-time around such a centre, thus:

$$ds^2 = \left(\cos^2\sigma - \frac{2L}{\Re \sin\sigma}\right) c^2 dt^2 - \frac{1}{g_4} dr^2 \\ - \Re^2 \sin^2\sigma \,(d\phi^2 + \sin^2\phi \, d\theta^2). \quad (102)$$

To consider the motion of a free particle in this field, develop the general equation of geodesics, $\delta \int ds = 0$, by the Lagrangean method. Since the variation of ϕ yields $d\phi/ds = 0$, we may conveniently put $\phi = \pi/2$, as on previous occasions. The variation of t and θ gives

$$\left(\cos^2\sigma - \frac{2L}{\Re \sin\sigma}\right) c\dot{t} = k, \quad \Re^2 \sin^2\sigma \dot{\theta} = p, \quad (103)$$

where k, p are constants, and these first integrals, together with the expression (102) itself or

$$\left(\cos^2\sigma - \frac{2L}{\Re \sin\sigma}\right) c^2 \dot{t}^2 - \frac{\cos^2\sigma}{g_4} \dot{r}^2 - \Re^2 \sin^2\sigma \dot{\theta}^2 = 1,$$

give the equations of motion

$$\left.\begin{aligned} \frac{1}{c}\frac{d\theta}{dt} &= \frac{p g_4}{k \Re^2 \sin^2\sigma} \\ \frac{1}{c}\frac{dr}{dt} &= \frac{g_4}{\cos\sigma}\left[1 - \frac{g_4}{k^2} - \frac{p^2}{k^2\Re^2}\frac{g_4}{\sin^2\sigma}\right]^{\frac{1}{2}} \end{aligned}\right\}, \quad (104)$$

where $g_4 = \cos^2\sigma - 2L/R \sin\sigma$. The equation of the orbit is

$$\frac{dr}{d\theta} = \frac{k\Re^2}{p}\frac{\sin^2\sigma}{\cos\sigma}\left[1 - \frac{g_4}{k^2} - \frac{p^2}{k^2\Re^2}\frac{g_4}{\sin^2\sigma}\right]^{\frac{1}{2}}$$

or, with the new variable

$$u = 1/\mathfrak{R} \sin \sigma, \qquad g_4 = 1 - \mathfrak{R}^{-2} u^{-2} - 2Lu,$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{p^2} - \frac{g_4}{p^2}(1 + p^2 u^2), \quad . \quad . \quad (105)$$

whence also

$$\frac{d^2 u}{d\theta^2} + u = \frac{L}{p^2} + 3Lu^2 - \frac{1}{p^2 \mathfrak{R}^2 u^3}. \quad . \quad (105')$$

For the motion of the actual planets about our sun as mass-centre this equation would lead to practically the same results as that for $\mathfrak{R} = \infty$. In fact, working out the approximate solution, one finds that superposed upon the famous Einsteinian effect (secular motion of the perihelion, per revolution), $\delta \varpi = 6\pi L^2/p^2$, there is a perihelion motion due to the finite curvature radius, which for each member of the solar system is, with $\mathfrak{R}$ of the order 10^{12} astr. units, but an exceedingly small fraction of $6\pi L^2/p^2$, much too minute to be ever observed.* It would remain imperceptible even for $\mathfrak{R} \doteq 10^9$, which, of course, is out of the question.

Such being the case, there is no need of discussing the implications of the orbit equation (105) in its full generality.

Some interest, however, is (with a view to interstellar instead of planetary distances) attached to the particular case of *circular* orbits.

We know that, in classical celestial mechanics, a circular orbit of any size, properly related to its

* Cf. Note 5.

period (Kepler's third law), is possible. Not so, however, in the isotropic, curved, world we are now considering. In fact, remembering the apparently centrifugal acceleration (active up to $\sigma = \pi/4$), the gravitation centre, e.g. a star, may be expected to lose its grip upon the moving particle, which may as well be another star, provided its distance from the centre reaches a certain limit, and if this is exceeded, the particle may desert the centre for ever. In brief. there should be an upper limit to the diameter of a circular or in fact any closed orbit. And the familiar Keplerian law relating it to the period of revolution can be expected to be replaced by some more complicated law. Such also turns out to be the case. In fact, the necessary condition for a circular orbit, $dr/dt = 0$, is, by (105) or by the second of (104), either

$$g_4 = 0, \quad \text{that is,} \quad \cos^2\sigma - 2L/\mathfrak{R}\sin\sigma = 0 \qquad (a)$$

or

$$g_4(1 + p^2u^2) = k^2. \quad . \quad . \quad . \quad . \qquad (b)$$

The first alternative is, as we saw before, satisfied by the huge radius (up to higher powers of $L/\mathfrak{R}$)

$$\sigma = \frac{\pi}{2} - \sqrt{\frac{2L}{R}}\left(1 + \frac{5}{6}\frac{L}{R}\right),$$

i.e. by an orbit almost touching the polar of the mass-centre, and need not, therefore, detain us any further.

The second alternative leads to a quintic for u in terms of k, p. What interests us, however, is not u

as a function of k, p but the period of revolution, say τ, in terms of the radius r of the orbit, or of its function $u = 1/\Re \sin\sigma$. Now, for a circular orbit we have from (102), with $\phi = \pi/2$, and $d\theta/dt = 2\pi/\tau$,

$$\frac{d}{du}\frac{\partial \dot{s}}{\partial \dot{\sigma}} = \frac{\partial \dot{s}}{\partial \sigma}, \qquad g_4 c\frac{dt}{ds} = k, \qquad \Re^2 \sin^2\sigma \frac{d\theta}{ds} = p. \quad (c)$$

From the last two equations we have, for the period,

$$\tau = 2\pi \frac{k}{p}\frac{\Re^2 \sin^2\sigma}{cg_4}, \tag{d}$$

and, by the first of (c), requiring $\dot{\sigma} = 0$ ($\sigma =$ const.), $d\theta/dt = cpg_4/kR^2 \sin^2\sigma$, which is identical with (d). Again, by the last two of (c),

$$\frac{p^2}{k^2} = \tfrac{1}{2}\frac{\Re^2}{g_4}\frac{dg_4}{d\sigma}\tan\sigma \,.\, \sin^2\sigma,$$

and, substituting in (d),

$$\frac{c^2\tau^2}{4\pi^2} = \frac{2R^2 \sin\sigma\cos\sigma}{g_4 dg_4/d\sigma},$$

or, after simple reductions,

$$\frac{4\pi^2(\Re\sin\sigma)^3}{\tau^2} = M\left[1 - \frac{\sin^3\sigma}{L/\Re}\right]\left[\cos^2\sigma - \frac{2L}{\Re\sin\sigma}\right]. \tag{106}$$

Such, then, is the required relation between the period τ and the radius $r = \Re\sigma$ of any circular orbit in the isotropic world. For $\Re = \infty$ this reduces at once to

$$\frac{4\pi^2 r^3}{\tau^2} = M,$$

the well-known expression of Kepler's third law, M being the mass of the central body in astronomical units.

Next, solving (106) for τ^2, and re-introducing the gravitation radius $L = M/c^2$ of the central body, we have (with T written again for $2\pi\mathfrak{R}/c$, the 'cosmic day')

$$\tau = \frac{T}{\sqrt{\left(\dfrac{L}{\mathfrak{R}\sin^3\sigma} - 1\right)g_4}} \quad . \quad . \quad . \quad (106')$$

Whence we see that (apart from the quasi-polar, $g_4 = 0$, already considered) *the greatest circular orbit possible*, for which also $\tau = \infty$, is that whose radius $r^* = \mathfrak{R}\sigma^*$ is determined by

$$\sin^3\sigma^* = \frac{L}{\mathfrak{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

If $r > r^*$, the period τ becomes imaginary; no such orbits are possible. Placed at any such distance from the centre, a particle would neither fall into nor circle around it, but escape (on a hyperbolically shaped path). At this distance r^*, which may be termed *critical*, the gravitational 'pull' is just compensated by the apparent centrifugal acceleration.† Approximately, but with more than sufficient accuracy even for a centre as massive as our whole

† Which, as we saw, may be imagined as due to a spin with the cosmic day $T = 2\pi\mathfrak{R}/c$ as period. No doubt, a formula of essentially the type of (106′) would also hold for a particle attracted by, say, an electrified central particle if both were placed on a turn-table.

galaxy ($L = \frac{1}{2}\ 10^{10}$ km. $= 33{\cdot}3$ astr. units), the definition (107) of the critical distance or critical radius of a mass Lc^2 can be written

$$r^* = (L\mathfrak{R}^2)^{\frac{1}{3}}. \qquad (107')$$

Formula (106′) can also be written, in terms of the critical radius,

$$\tau = \frac{T}{\sqrt{\dfrac{\sin^3\sigma^*}{\sin^3\sigma} - 1}} \doteq \frac{T}{\sqrt{\left(\dfrac{r^*}{r}\right)^3 - 1}} \qquad (106'')$$

It may be well to consider a few numerical instances. Thus, for the sun ($L = 1{\cdot}47$ km.), with $\mathfrak{R} = 37{,}600$ kiloparsecs or $1{\cdot}16 . 10^{21}$ km., the critical radius is

$$r^*_{\odot} = 4{\cdot}07 \text{ parsecs},$$

corresponding to a parallax $0''{\cdot}246$. A particle placed at rest at any point of this neutral sphere (so to call it) would stay there for ever, in absence of other masses, of course. This sphere is a good stretch beyond our nearest stellar neighbour, Proxima Centauri, placed at 1·31 parsec. If there were no other stars, the latter would revolve around the sun, and if its orbit were circular, then (neglecting the wobbling of the sun)† its period of revolution would be, by (106″), $0{\cdot}186\ T$, or about one-fifth of a cosmic day.

If $(\mathfrak{R})$ be the number of light-years contained in

† In our treatment of this subject the mass $m = lc^2$ of the 'particle' has been tacitly assumed to be negligible in comparison with that of the 'centre'. To take account of their finite mass-ratio, in a rough way, one would have to replace formula (107′) by

$$r^{*3} = (L+l)\,\mathfrak{R}^2.$$

$\mathfrak{R}$, the number of years contained in T will simply be 2π ($\mathfrak{R}$). Thus, with the above value of the curvature radius, $(\mathfrak{R}) = 3{\cdot}26 \,.\, 3{\cdot}76 \,.\, 10^7 = 1{\cdot}23 \,.\, 10^8$, and

$$T = 770 \text{ million years.}\dagger$$

The fictitious round trip of α Centauri about the sun would thus take 143 million years. But if that star were only three (or 3·1) times as distant, such a trip would never materialize.

The separation of *double stars* is, to the best of my knowledge, in all observed cases much smaller than the critical distance, $r^* = \left[\frac{M_1+M_2}{c^2}\mathfrak{R}^2\right]^{\frac{1}{3}}$. In fact, for Capella it is only 0·85 astr. units, and it scarcely ever exceeds a few astronomical units, while r^* will in general be of the order of ten parsecs. (At any rate the theoretical result in hand is not contradicted by observations on pairs of stars, or trios, &c., whose dynamical interdependence can actually be proved.)

To pass at once to a 'centre' of gigantic massivity, let us take the case of our whole galaxy. According to Kapteyn its mass is equal to $\frac{1}{3}10^{10}$ suns, which is equivalent to a gravitation radius

$$L_{galaxy} = \tfrac{1}{2}\,10^{10} \text{ km.}$$

The critical radius, which would correspond to this mass if it had a spherical distribution, turns out to be

$$r^*_{galaxy} = 1{\cdot}89 \,.\, 10^{17} \text{ km.} = 20{,}000 \text{ light-years,}$$

† Cf. end of Note 4.

that is to say, 4/3 times the actual minor and $7\frac{1}{2}$ times smaller than the major semi-axis of our galaxy, 15 and 150 thousand light-years respectively, as estimated by Shapley, and already mentioned. The stars near the rim of the galactic plane should thus be continually deserting us, or—as the writer put it some years ago—our home universe is, in relation to its mass, too inflated to be permanent. It resembles in this respect the ancient Roman Empire at the maximum of its territorial expansion and, in our own days, possibly (who would dare to predict it?) the British Empire. Most of the globular clusters, so thoroughly investigated by Shapley, can—on these principles—be attributed a high degree of permanency. And such also is the impression by looking at these beautiful globes on a clear night. They may, perhaps, be compared with Switzerland, which, in spite of powerful external attacks in olden times, remains undisrupted these eight centuries or so.

But political diagnosis and prognosis are not among the topics planned for this book.

Let us therefore conclude these numerical examples by stepping down to atomic and sub-atomic entities.

The gravitation radius of a gramme of mass is

$$L_{gr} = 7{\cdot}40 \,.\, 10^{-29}\,\text{cm.},$$

which follows at once from the gravitation constant $6{\cdot}66 \,.\, 10^{-8}$. And since the mass of a hydrogen atom is $1{\cdot}66 \,.\, 10^{-24}$ gr., and that of its positive nucleus, the

proton, only by about 1/1845th smaller, the critical radius of such an atom or a proton is

$$r^*_{proton} = 1{\cdot}18 \text{ cm.}$$

The 'radius' of an electron (i.e. the length $\frac{2}{3}\, e^2/mc^2$) is $1{\cdot}87 \,.\, 10^{-13}$ cm., and therefore that of a proton,† $1{\cdot}01 \,.\, 10^{-16}$ cm. Thus, the critical radius r^* of a proton is very nearly 10^{16} times as large as its geometrical radius. If even the proton were a galaxy of yet smaller entities, it would have as a system to be attributed an extremely stiff permanency. At any rate its (purely gravitational) pull should cease entirely at a distance of little more than one centimetre, for an observer placed on the proton, that is.

These examples will suffice to illustrate the period and critical-distance formulae, and since we dared to apply them, though derived for a single point-mass, to galaxies or systems of many stars, we will better turn now to supplying the proof of the legitimacy of such an application.

Now, as the dimensions of even a stellar system such as our own or the extra-galactic nebulae, are small fractions of the world radius $\mathfrak{R}$ (whether this runs into thirteen or twelve figures in a.u.), we can work out the required formulae under the assumption of small $\sigma = r/\mathfrak{R}$. Thus, as was mentioned before, we can represent the resultant centrifugal tendency by $\omega^2 r$, where $\omega = c/\mathfrak{R}$, and simply superpose this

† Assuming that its mass is again of purely electro-magnetic origin.

upon the Newtonian gravitational attraction $-M/r^2$, of which the corresponding potential $M/r = c^2L/r$ enters in g_{44} through $2L/r$ or $2L/\mathfrak{R} \sin \sigma$. If we first consider a purely radial motion ruled by a single mass-centre M, then this approximate method gives

$$\frac{d^2r}{dt^2} = \omega^2 r - \frac{M}{r^2} = c^2\left(\frac{r}{\mathfrak{R}^2} - \frac{L}{r^2}\right),$$

whence the critical distance $r^* = (L\mathfrak{R}^2)^{\frac{1}{3}}$ follows immediately. The first integral,

$$\tfrac{1}{2}\beta^2 - \frac{L}{\mathfrak{R}\sigma} - \tfrac{1}{2}\sigma^2 = \text{const.},$$

is obtained at once, and can easily be discussed. This, however, may be left to the reader as a non-uninteresting exercise. The more general case of non-radial motion will, with x, y as Cartesian coordinates of the particle (whose orbit is manifestly contained in a plane, laid through O, the mass-centre, and the initial line of motion), be characterized by the two differential equations

$$\frac{1}{c^2}\frac{d^2x}{dt^2} = \frac{x}{\mathfrak{R}^2} - \frac{Lx}{r^3},$$

$$\frac{1}{c^2}\frac{d^2y}{dt^2} = \frac{y}{\mathfrak{R}^2} - \frac{Ly}{r^3},$$

which admit the well-known first integrals of 'vis-viva' and of areas, leading easily to a complete solution. But even this need not detain us here. For, after these brief preliminaries, we are eager to

contemplate the case of a galaxy of stars, schematized down to a spherical and uniform distribution of a large number of equal point-masses (stars) filling out a globe of radius a. If $M = c^2L$ be the total mass of such a galaxy, the equation of motion of any of its constituent stars are

$$\frac{1}{c^2}\frac{d^2x}{dt^2} = \frac{x}{\Re^2} - \frac{Lx}{a^3}, \text{ \&c.,}$$

i.e.

$$\frac{1}{c^2}\frac{d^2x}{dt^2} = -Q^2x, \quad \frac{1}{c^2}\frac{d^2y}{dt^2} = -Q^2y, \quad . \quad (108)$$

where

$$Q^2 = \frac{L}{a^3} - \frac{1}{\Re^2} = \frac{1}{\Re^2}\left(\frac{r^{*3}}{a^3} - 1\right), . \quad (108\,a)$$

r^* being, by (107′), the critical radius of the total mass of the galaxy. That in this case, also, the orbit of every star is a plane curve is manifest. Its equation follows at once from (108), whose general solution, with a proper choice of the origin of time reckoning and the orientation of the coordinate axes, can be written

$$x = A\cos(cQt), \quad y = B\sin(cQt). . \quad (109)$$

Thus if $Q^2 > 0$, i.e. $a < r^*$, the orbit of every star is an ellipse, concentric with the spherical galaxy,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.$$

Its semi-axes, A, B, will vary, of course, from star to star in size and direction. The period $\Theta = 2\pi/cQ$

of revolution, however, will be common to all the stars, or a property of the galaxy as a whole. By (108a), its length will be, again in terms of the cosmic day $T = 2\pi\Re/c$,

$$\Theta = T/\sqrt{(r^*/a)^3 - 1}. \quad . \quad . \quad . \quad (110)$$

So long as $a < r^*$, this galactic period will be real, and the longer the closer the semi-diameter of the galaxy approaches its critical radius. And all the constituent stars, some describing swiftly comparatively huge ellipses, some moving lazily on small elliptic orbits, will remain in the system for ever, nay its configuration will repeat itself exactly after the lapse of each period Θ.

If $a = r^*$, we have $\Theta = \infty$, and, as we see directly from (108), each star will move uniformly on a straight line, utterly asocial, as it were, that is, disregarding the presence of all other members of this galactic community. Thus, if their instantaneous velocities are haphazard in size and direction, the crowd will sooner or later be scattered to the four winds as, in spite of all visual appearances, there is nothing to hold them together. And the more so, if the galaxy has swollen up beyond that critical radius of its mass total, the orbits then degenerating, by (109), into hyperbolic branches. Is in the wording of these results an illusion implied to communities of humans, a score or so of Nation-Galaxies, now stiffly coherent, now torn asunder, not only by foreign interference but as often by internal strife, with the

greed and lust of power of the 'upper thousand' to urge the territorial expansion (growing a), and a huge majority of what some modern sociologists* refer to as the 'international mob', but too eager to profit by every tumult, to tear down all social bonds, to set out on world-wide vagabondage, and mingling with deserters of other galaxies, sweep down every civilization, like the sand-laden winds of Sahara obliterate every hillock or oasis casually formed of stray particles through long years of slow and patient toil? I prefer to leave this question to my readers, the more so as I am the last man to desire to undermine their possibly strong confidence in a steady evolution and betterment of assemblages, both human and celestial, in an ever-mounting 'progress', that is, as opposed to the hurrican-like 'cycloism' which is (not without weighty reasons) advocated by the aforesaid group of social philosophers. At any rate, some time spent now and then on such comparisons will not be entirely lost, provided that these are tempered by caution and self-criticism, and not poisoned by passion.

Returning to formula (110) for the period of a spherical galaxy, let us work it out for, at least, two actual (though somewhat schematically simplified) cases, to wit, that of our own Milky Way and one of

* Namely, the advocates of the theory of *social cycloism*. See a very interesting booklet on *Sozialphilosophie*, by Prof. L. Gumplowicz, Innsbruck, 1910.

the well-studied globular clusters, N.G.C. 6205.† As we saw before, the critical radius of our galaxy is $r^* = 20{,}000$ light-years. With regard to its shape, it resembles rather a 'flat' modern watch. Yet we may, not without some instructiveness as to its dynamical behaviour, replace it for the purpose in hand by an equivoluminous sphere of uniform mass density of stars, and thus (using Shapley's dimensions, already quoted) put

$$a = (1{\cdot}5^2 \,.\, 10^{10} \,.\, 1{\cdot}5 \,.\, 10^4)^{\frac{1}{3}} = 69{,}600 \text{ light-years.}$$

Now, this semi-diameter is about $3\frac{1}{2}$ times the critical radius of the galaxy. In fine, if our theory covers correctly the actual spacetime, the Milky Way is about $3\frac{1}{2}$ times (linearly, or about 43 times volumetrically) *too large* to be permanent. If the theory is right, this galaxy is just a casual gathering of intergalactic vagabonds without mutual sympathy or solidarity, as it were, and constantly deserting us, every one of them (and especially thus at the outskirts of the Milky Way), in order to settle down, perhaps after millions of years of free vagabondage, in some other, more stoutly organized galaxy already established, or else in conjunction with similar specimens of intergalactic wanderers to form some new stellar system of lesser or greater 'permanency'.

It may be well to mention in this connexion that the Swedish astronomer, Prof. Charlier, in whose

† The same object that appears in Table I.

(classical) equations the 'centrifugal' term $\omega^2 r$ was absent, and whose formula for the period of a galaxy was, accordingly,

$$\Theta = 2\pi\sqrt{\frac{a^3}{M}} \quad . \quad . \quad . \quad \text{(Charlier)}$$

[i.e. our (110) for $\Re = \infty$], never suspected the possibly ephemeral nature of the Milky Way. On the contrary, in a most attractive booklet,* entitled 'How an Infinite World may be built up by C. V. L. Charlier', in which he develops in a masterly way Lambert's fascinating concept of an endless hierarchy of galaxies, galaxies of galaxies, &c., he closes a certain section by emphasizing the result that 'after this time [the period of the galaxy] the Galaxy resumes the same constitution and appearance', that is to say, not only our own but also all other galaxies (of the first and of any higher order) considered in his work. In fact, by Charlier's formula just quoted, or

$$\Theta = \frac{2\pi}{c}a^{3/2}L^{-\frac{1}{2}}, \text{ and } L = \tfrac{1}{2}\,10^{10}\,\text{km.} = 5{\cdot}29\,.\,10^{-4}\ \text{l.y.},$$

$a = 69{,}600$ light-years (as above), one would have

$$\Theta_{\Re=\infty} = 5{\cdot}02\,.\,10^9 \text{ years,}\dagger$$

* Reprinted from *Arkiv för Mathematik, Astronomi och Fysik*, xvi, No. 22, Stockholm, 1922.

† Charlier himself (loc. cit.) having assumed $M = 10^9$ suns, i.e. $L = \frac{3}{2}10^9$ km., and $a = 1000$ 'Siriometers' $= 10^9$ a.u. $= 15{,}800$ l.y., finds $\Theta = 10^9$ years.

which is, comparatively speaking, a short period. One would have to accept it, of course, only if one gave up the hope of ever setting an upper limit to the curvature radius $\mathfrak{R}$ of our spacetime. Charlier's paper has been mentioned here only in view of the stimulating nature of Lambert's original idea and its masterly treatment by the great Swedish astronomer. But an exposition of this theory (taken up very actively also by F. Selety and in the United States of America fervently defended by my friend W. D. MacMillan) at some length would not answer the purpose of the present book. For while we are here mainly concerned with making some finite value of $\mathfrak{R}$ plausible and developing its chief implications, the Neo-Lambertian, hierarchical, cosmology is most essentially constructed with a view to an '*infinite world*'. (A condensed statement of Charlier's work is given in Appendix H of my book on Relativity, though more pleasure will be derived from reading Charlier's own paper, just quoted, and Selety's thorough investigation in *Ann. der Physik*, lxviii, pp. 281–334, 1922, which was followed by a long polemic between the latter author and Einstein.)

Passing to the second of the promised examples, the globular cluster N.G.C. 6205 has, if I may judge from a good Mount Wilson slide, a radius $a = 79$ light-years. From an enlarged photographic print of this slide I find by actual counting the number of distinct stars contained in the cluster to be not less

than 11,900. Thus, if each of these were as massive as our sun, we should have

$$L = 17{,}500 \text{ km.} = 1{\cdot}85 \,.\, 10^{-9} \text{ light-years,}$$

and the critical radius

$$r^*_{6205} = [(1{\cdot}23 \,.\, 10^8)^2 \,.\, 1{\cdot}85 \,.\, 10^{-9}]^{\frac{1}{3}} = 304 \text{ light-years,}$$

that is, almost four times the semi-diameter. It would follow that the necessary condition of permanency (periodical recurrency of configuration) is amply satisfied for this globular cluster. And its period would be, by (110),

$$\Theta_{6205} = \frac{\text{cosmic day}}{7{\cdot}46}$$

or 103 million years, which in cosmology is a rather short time.

The necessary condition of permanency of a spherical galaxy, $a < r^*$ can be put in yet another interesting form. The distribution of mass within the galaxy being, macroscopically, uniform, let ρ be its density in astronomical measure. Then, neglecting higher powers of $a/\Re$,

$$r^{*3} = \Re^2 L = \frac{4\pi\Re^2}{3} \frac{\rho a^3}{c^2},$$

and the condition becomes

$$\rho > \frac{3c^2}{4\pi\Re^2} \,. \quad . \quad . \quad . \quad . \quad (111)$$

Thus, *the critical density* is a universal constant,

$$\rho^* = \frac{3c^2}{4\pi\Re^2} = \frac{3\pi}{T^2} = \frac{3}{4\pi}\omega^2, \quad . \quad . \quad (112)$$

T being the cosmic day, and $\omega = c/\mathfrak{R}$ the universal spinning velocity. If the density of a galaxy of stars or, for that matter, of a swarm of molecules, no matter how small or large in extension, falls below this critical value ρ^*, the stars or the molecules will desert the system.† Since M/c^2 is a length, ρ^*/c^2 should be a reciprocal area, and such, by (112), it is, namely,

$$\rho^*/c^2 = \frac{3}{4\pi} \times \text{curvature of elliptic space.}$$

If the last found value (98_1) of the curvature radius, i.e. $\mathfrak{R} = 1{\cdot}16\,.\,10^{21}$ km., is adopted,

$$\frac{\rho^*}{c^2} = 1{\cdot}93\,.\,10^{-43}\,\text{km.}^{-2}.$$

To express this result in more familiar terms, let us recall that the sun's gravitation radius (M/c^2) is $1{\cdot}47$ km., so that the density of one sun per cubic kilometre is

$$\frac{\odot}{\text{km.}^3} = \frac{1{\cdot}47\,\text{km.}}{\text{km.}^3} = 1{\cdot}47\,\text{km.}^{-2}.$$

Consequently, the critical density is

$$1{\cdot}31\,.\,10^{-43} \quad \odot/\text{km.}^3$$

or *one sun* per $7{\cdot}63\,.\,10^{42}$ km.^{3}, i.e. per 260 cubic parsecs, or also, since $\odot = 1{\cdot}99\,.\,10^{33}$ gr.,

$$2{\cdot}61\,.\,10^{-25}\,\text{gr. cm.}^{-3}, \quad . \quad . \quad (112_1)$$

or, finally, three protons per 19 cubic centimetres.

† The same 'critical density' belongs also to a *continuous* distribution of matter, say a fluid globe. Cf. Note 6.

Small as this critical mass density may seem when measured by ordinary standards, the mean density of our galactic system is more than forty times smaller, and this system is therefore doomed to dissolution. In fact, as we saw, it is (linearly) 3½ times too big to claim permanency.

Notice in passing that Dr. Hubble's mean space-density of granular distribution of matter (with nebulae as grains), quoted above on p. 97, is about two million times smaller than our critical density. Thus a uniform galaxy of galaxies (Hubble's nebulae, each of mass $2{\cdot}6 \, . \, 10^8 \odot$) of such a small density of distribution, $1{\cdot}5 \, . \, 10^{-31}$ gr. /cm.^{3} = $9{\cdot}10^{-18}$ nebula/pars.^{3}, if placed in our isotropic world,* would at once be dissolved.

We have now sufficiently learned what to expect of a finite isotropic world, whether its curvature radius be 36 million parsecs or a little smaller. Curiously enough, while, on the one hand, it imprisons us for ever, atoms, humans, stars, galaxies, and all, in a finite though huge re-entrant cage (of some 10^{64} cubic kilometres), it does not, on the other hand, encourage the protracted existence of huge and comparatively flimsy, non-massive, congregations or empires, compensating—as it were—the loss of universal freedom by copious opportunities for an individual escape from many an incoherent system.

* Of course, not all of Hubble's 'normal' nebulae, $3{\cdot}5 \, . \, 10^{14}$ in number (in his antipodal, and $1{\cdot}75 \, . \, 10^{14}$ in polar space), could thus be placed in our space, which is (linearly) 750 times smaller than his. This, however, does not affect the gist of our argument.

The main object of this Essay is now materially exhausted. Some supplementary information and discussion had better be relegated to the appended Miscellaneous Notes. Of these some will have to be written in the course of printing of the main text, so as to enable the author to bring matters up to date, and especially to utilize, for a more accurate determination of $\mathfrak{R}$, all such further star data as may meanwhile be forthcoming.

[*Added in proof reading.*] The reader's attention should especially be called to Note 8, where a new determination of $\mathfrak{R}$ is given. This is based upon a huge number of star data (459 stars), comprising besides the radial also the transversal velocities. The resulting radius, 4.10^{11} a.u., is thus independent of any arbitrary assumptions about the star-constants r_0 and can claim a much higher degree of reliability than the values found in the main text, the more so as it agrees nearly enough with the results obtained by the same method in Notes 3 and 4 from the Cepheids and the O-stars. Some of the more interesting derived magnitudes, as the cosmic day and the critical density, are re-calculated, in accordance with the new $\mathfrak{R}$-value, at the end of Note 8. The former is reduced and the latter is increased. Thus e.g. the instability of our own galaxy is, of course, still more pronounced than it appears from the preceding discussion.

MISCELLANEOUS NOTES

1. Space-Curvature and World-Curvature.

It may be well to draw the reader's attention to the fact that while the curvature of the three-space, as section of the four-dimensional isotropic spacetime or de Sitter's world, i.e. of the elliptic space used throughout the book, is positive, namely $\mathfrak{R}^{-2}$, the *world-curvature* itself is *negative*, to wit, as in (42), p. 61,

$$K = -\frac{1}{12}(\text{invariant of curvature tensor}) = -\frac{1}{\mathfrak{R}^2},$$

similarly as for an ordinary surface, the Gaussian curvature $K = -\frac{1}{2}R$, and for any isotropic n-fold,

$$K = -\frac{R}{n(n-1)}, \quad R = g^{\iota\kappa} R_{\iota\kappa}.$$

In fine, the de Sitter world is a manifold of constant *negative* curvature.

An isotropic world of constant positive curvature $K = +\mathfrak{R}^{-2}$ (which nobody seems to have contemplated) would have for its line-element, instead of (59),

$$ds^2 = \cosh^2\sigma \,.\, c^2 dt^2 - [dr^2 + \mathfrak{R}^2 \sinh^2\sigma\,(d\phi^2 + \sin^2\phi\, d\theta^2)], \qquad (a)$$

$\mathfrak{R}$ being simply replaced by $i\mathfrak{R}$. The bracketed expression is the element of a Lobatchevskyan three-space which, historically, is the oldest non-Euclidean space known and studied (just about 100 years old). It is, of course, infinite. The world (a), therefore, of which this is a section normal to the t-axis, might appeal to some astronomers who hate to feel hedged in in a finite volume. Yet it is scarcely worth contemplating, since it would give, apart from niceties, in the case of, say, purely radial motion, $D^2 = v_0^2/c^2 - \sigma^2$ instead of $v_0^2/c^2 + \sigma^2$, that is, a Doppler effect *decreasing* with distance which certainly is not the case. In fact, even though the numerical value of the correlation coefficient between $|D|$ and σ is not quite so great as one might desire it to be, its sign is decidedly *positive*, signifying a systematic increase of $|D|$ with r, which is seen also at a glance from Fig. 11. The world (a) may thus have but a purely theoretical interest.

2. A Rejoinder to Prof. H. Weyl's Criticism.*

In his 'Observations' (*Phil. Mag.*, vol. xlviii, August 1924, pp. 348–9) on my paper on the 'Determination of the Curvature Invariant of Spacetime' (*Phil. Mag.*, vol. xlvii, pp. 907–17), Prof. Weyl of Zurich defends his position against some remarks made incidentally in the last-said paper.

Before attacking the main point, concerning the world-lines of the stars, Weyl, in defending his 'tan' formula,

$$D = +\tan(r/\mathfrak{R}),$$

which even with his different meaning of 'r' can be readily shown to be erroneous, expresses the opinion that the coordinate r in my formula [(88′) of this book] 'signifies *a very artificial quantity*—namely, the distance of the star from the observer at the moment of observation, but in the static space of the star'.

In what sense my r is more 'artificial' than that relating to the static space of the observer will remain a secret known only to Weyl himself. Again, his remark that this r may in some instances be complex (imaginary) strikes me as quite irrelevant. If r_e be, in the same static space of the star, the distance at the moment of *emission* (while r refers to that of observation), we have, e.g. in the sub-case $v_0 = 0$ and for receding radial motion, the relation

$$\sin\frac{r}{\mathfrak{R}} = \frac{\tan(r_e/\mathfrak{R})}{1-\tan(r_e/\mathfrak{R})},$$

so that for $r_e > \mathfrak{R}\arctan\frac{1}{2}$, r does, in fact, cease to be real. But this is not a reason against the use of such a coordinate. This means simply that light signals never reach an observer thus situated relatively to the source at the moment of emission, which is only one more interesting peculiarity of the curved isotropic spacetime, foreign to the homaloidal one of Einstein-Minkowski. It would, moreover, reassert itself, giving in certain cases an imaginary distance, also if the static space of the *observer* were

* This note was materially drafted as a 'Rejoinder' to H. Weyl as long ago as 7 Sept. 1924 but, in the midst of other occupations, was not published until now.

used. But there is scarcely any need for insisting further upon this point.

Next, while I referred to Weyl's statement regarding the world-lines of the stars (viz. that they form a pencil diverging into the future—equivalent to his 'universal scattering tendency') as a 'more or less disguised and gratuitous hypothesis', I certainly had no intention of imputing that Prof. Weyl has 'disguised' it deliberately, but only that it does not (without a painstaking scrutiny), especially in his book *Raum-Zeit-Materie* (Appendix, p. 322, 5th ed.), appear as a mere assumption.* Even now, though Weyl states clearly its hypothetical character, he obscures this somewhat by declaring that it is *necessary* to make such an assumption.

The important thing is that Weyl's hypothesis (that 'the worldlines of the stars belong to a pencil of ∞^3 geodesics diverging towards the future') is much too narrow to cover the facts of stellar astronomy.

A very strange impression, I might almost say, one of deliberate unfairness, will every unbiased reader receive from Dr. Weyl's last (third) 'observation', which runs thus:

'Curiously enough, Dr. Silberstein at the end of his articles *uses exactly the same assumption as a basis, the only difference* being that he *adds* to my group of worldlines, which diverge into the future, that which results from it through the interchange of past and future (double sign). That, I must confess, appears quite abstruse to me.'

Now, I do *not* use 'the same assumption'; nor do I 'add' to his group of diverging (i.e. radially receding), that of converging (i.e. radially approaching) stars, but exactly, making *no assumption at all* about the sense of motion and orientation of orbits, consider the most general case of stars crossing the space in all possible directions (w.r. to the observer) and treating the radial motions, both receding and approaching, as particular cases. In general I work with the formula relating to oblique motions (v. *supra*). And the need for such generality (instead of trying, as Weyl does, to force upon the heavenly host of bodies a radial

* Cf. also my *Theory of Relativity*, 1924, footnote to p. 515.

divergence) is manifest to everybody who takes the pain of inspecting the observed radial velocities and proper motions of actual stars or stellar systems, instead of indulging in purely mathematical conjectures and speculations.

3. Correlation between Radial Velocity and Distance supported by a Group of Cepheid Variables. The corresponding $\mathfrak{R}$-estimate.

The observational material relating to twenty-nine Cepheid Variables, which was kindly furnished to me by Dr. Ralph E. Wilson, of Albany,* at the end of 1924, and which at that time, perhaps too rashly, was shifted aside by me as not very promising (in view of the comparatively small distances of these objects), has, after a recent scrutiny, turned out to afford very strong evidence for the correlation between velocity and distance. These star data, moreover, offer the additional interest of containing also measurements of proper motions (transversal velocities), not obtainable in the case of the more distant objects, the globular clusters, Clouds, and spiral nebulae, dealt with in the main text of this book, Part V.

As a matter of fact, this source of evidence is as good as, if not better than, that afforded by the clusters. Yet, to avoid rearrangement of Part V, which was all ready for printing before I noticed this precious property of the Cepheids, this little investigation has been relegated to the 'Notes'. (And exactly the same circumstances induced me to treat similarly Prof. Plaskett's data for the O-stars, which will be analysed in the next note, and the data for 459 other stars to be utilized in a further note.)

The set of data, for twenty-nine Cepheid Variables, taken from Wilson's paper, and supplemented by the column of distances r, in hectoparsecs, according to Shapley's measurements,† and the (linear) transversal velocities, $v_t = r\mu$ (dist. $\times$ proper motion), and the resultant velocities v, is given in Table IV. All velocities, v_t, v, and the radial one v_r (absol. value), are in km./sec. Two

* Through a copy of his paper, *Astr. Journal*, 1924, No. 821, p. 40, and a number of private explanations thereto.

† *Mount Wilson Contrib.*, No. 153, 1918.

Cepheids, showing comparatively exorbitant radial velocities, and one more (β Cephei), just to leave an even number to be divided in the sequel into two equinumerous sub-groups, are listed separately in the lower part of the Table. It will be noted that for two other Cepheids, SW Dra and SV Mus, the proper motions are not available.

TABLE IV. *Cepheid Variables*

Object	r in 100 pars.	μ in 1″	v_t	v_r	v
1. α U Mi	0·6	0·046	13·1	6·4	14·6
2. δ Ceph	1·8	0·011	9·4	3·9	10·2
3. η Aql	2·2	0·013	13·6	2·5	13·8
4. SU Cas	2·6	0·014	17·3	1·5	17·4
5. RT Aur	2·6	0·024	29·6	10·9	32·0
6. ζ Gem	2·8	0·010	13·3	6·3	15·0
7. X Sgr	2·9	0·018	24·7	2·1	25·0
8. W Sgr	3·0	0·011	15·6	17·7	24·0
9. κ Pav	3·1	0·017	25·0	33·2	41·0
10. RR Lyr	3·3	0·223	349·0	50·0	353·0
11. T Vul	3·6	0·000	0·0	16·0	16·0
12. Y Sgr	4·2	0·022	43·7	18·1	47·0
13. SU Cyg	4·8	0·024	54·6	15·9	57·0
14. RTνA	5·5	0·026	66·8	24·3	71·0
15. S Sge	5·5	0·006	15·6	10·2	19·0
16. SZ Tau	5·5	0·022	57·4	16·2	60·0
17. TV Cas	6·6 (?)	0·080	250·0	19·3	251·0
18. STrA	7·2	0·009	3·1	0·6	3·2
19. l Car	7·6	0·021	75·6	10·4	76·3
20. RV Sco	7·9	0·028	105·0	19·1	107·0
21. SV Mus	9·5	—	—	6·7	—
22. Y Oph	11·5	0·003	16·4	12·2	20·0
23. X Cyg	12·0	0·018	102·0	27·3	107·0
24. SW Dra	12·0	—	—	74·0	—
25. RS Boo	13·2	0·014	88·0	51·0	101·5
26. T Mon	15·2	0·030	216·0	5·1	216·1
27. β Ceph	0·6	0·019	5·4	0·0	5·4
28. SV Dra	8·3	0·131	515·0	193·0	550·0
29. XZ Cyg	10·0	0·114	540·0	196·0	574·0

For the correlation coefficient k, between the radial velocities and distances we have, for the twenty-seven Cepheids, viz. 1 to 27, in symbols used in Part V,

$$\Sigma \xi^2 = 825 \cdot 1, \quad \Sigma \eta^2 = 9579, \quad \Sigma \xi \eta = +1895,$$

whence

$$k = k(v_r, r) = 0 \cdot 674 \pm 0 \cdot 0684, \quad . \quad . \quad . \quad (a)$$

that is, very nearly *ten* (9·85) times its P.E. Needless to say, this is a very strong evidence for the correlation predicted and discussed in Part V, and thus also an excellent support of the isotropic spacetime or (as may be convenient to call the corresponding theory) *the isotropic cosmology.* This piece of evidence is much stronger than that offered by the globular clusters and the Magellanic Clouds (and the thirty-eight spiral nebulae). Not only is k/P.E. much greater, but r and v_r are known for the Cepheids with considerably greater accuracy, and this outweighs the much smaller distance (just reaching 1,500 parsecs) of these stars.

Thus far the radial velocities. Before utilizing, numerically, *the transversal,* and thus also the resultant ones, it is necessary to find out whether v_t also can be expected, on the whole, to increase with distance.

Now, by (91 a),

$$v_t = \frac{v_0 r_0}{r} \cos^2 \lambda,$$

where λ is the distance of the star from its perihelion (P) with $\mathfrak{R}$ as unit. Now contemplating the (quasi-) geodesic triangle $\odot$ P star, we have, by elementary elliptic trigonometry, which is identical with ordinary spherical trigonometry on a sphere of radius $\mathfrak{R}$,

$$\cos^2 \lambda = \cos^2 \sigma / \cos^2 \sigma_0, \quad \sin^2 \sigma_0 = \tan^2 \lambda / \tan^2 \theta,$$

where θ is the angle between the radius vector and the line of apses. Whence, after some simple reductions,

$$\cos^2 \lambda = \cos^2 \theta + \sin^2 \theta \, . \cos^2 \sigma,$$

and for small σ,

$$\cos^2 \lambda = 1 - \sigma^2 . \sin^2 \theta.$$

Thus,

$$v = \frac{v_0 r_0}{r}(1-\sigma^2 . \sin^2\theta) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

This differs from the classical result $v_t = v_0 r_0/r$ (corresponding to $\mathfrak{R} = \infty$) only by the term containing σ^2. In other words, while the 'distance effect' upon the radial velocity was of the first, that upon the transversal velocity is one of *the second order* in $r/\mathfrak{R}$, and is thus much too small to be ever detected.

At the same time we see that, to all purposes,

$$v_0 r_0 = r v_t , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

so that, if r and v_t are known, one of the two unknown star-constants, say r_0, can readily be eliminated. Thus our formulae for the radial and the resultant velocity become

$$v_r^2 = \left(1 - \frac{v_t^2}{v_0^2}\right)(v_0^2 + c^2\sigma^2)$$

and

$$v^2 = v_0^2 + c^2\sigma^2\left(1 - \frac{v_t^2}{v_0^2}\right);$$

in the last term v_0 can, roughly, be replaced by v, which in the final determination will not seriously affect the order of $\mathfrak{R}$. Thus,

$$v^2 = v_0^2 + \left(1 - \frac{v_t^2}{v^2}\right) c^2\sigma^2 = v_0^2 + \frac{v_r^2}{v^2} c^2\sigma^2 . \quad . \quad . \quad . \quad (c)$$

To the same order of approximation we have, as in ordinary (homaloidal) conditions,

$$\cos\theta = \frac{r_0}{r} = \frac{v_t}{v} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Formula (c) enables us to determine the curvature radius by the two-groups method *without* making any assumption about the ratio r_0/r or the corresponding angle θ, and this is a considerable advantage over the treatment adopted in Part V. In fact, if we assume only that for the two groups (1, 2) the mean $\overline{v_0^2}$ has the same value, (c) gives at once

$$\mathfrak{R}^2 = c^2 \Delta \overline{\frac{r^2 v_r^2}{v^2}} \Big/ \Delta \overline{v^2} , \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

where

$$\Delta x = x_2 - x_1 .$$

Applying this formula to two groups each of *twelve* Cepheids, viz. 1 to 12 and 13 to 26 (Table IV), with the omission of 21 and 24, for which the μ and therefore also the v are not available, we have, in *squared* hectoparsecs and km./sec.,

$$12\,\Delta\,\overline{\frac{r^2 v_r^2}{v^2}} = 122{\cdot}38 - 53{\cdot}29 = 69{\cdot}09,$$

$$12\,\Delta\,\overline{v^2} = 160{,}930 - 131{,}720 = 29{,}210.$$

Thus,

$$\left.\begin{aligned}\Re &= 1{\cdot}46\,.\,10^6 \text{ parsec}\\ &= 3{\cdot}01\,.\,10^{11}\text{ a.u.}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (f)$$

This value of the radius is about twenty-five times smaller than that, (98_1), derived from the clusters and clouds. Yet it is not to be despised. Nay, inasmuch as the Cepheids have shown a much stronger correlation and as the method applied to them (not prejudicing r_0/r) is less arbitrary than the previous one, the last-found curvature radius may deserve to be given a greater weight than the previous $7{\cdot}4\,.\,10^{12}$ a.u. And the more so, as another group of stars, to be treated in the next Note, happens to yield very much the same radius.

One might think that such a space would be too small to shelter the spiral Messier 33 (N.G.C. 598). But such is not the case. In fact, even if we accept Dr. Hubble's distance estimate 275,000 parsecs or $5{\cdot}6\,.\,10^{10}$ a.u., the largest distance possible in our space, $\frac{\pi}{2}(f) = 4{\cdot}7\,.\,10^{11}$, would still be over eight times greater than the distance of that spiral.

4. Correlation Coefficient and Curvature Radius from O-stars Data.

Professor J. S. Plaskett's memoir on *The O-type Stars** contains the radial velocities, proper motions, and (implicitly) the distances of forty-five stars, but of these only thirty-five are O-stars, the remaining ten, though listed together with the O's in Table XI, being B-stars. And as it has seemed advisable to deal

* *Publications of the Dominion Observatory, Victoria, B.C.*, vol. ii, No. 16, 1924, p. 328.

with a homogeneous material, we will consider here only those thirty-five stars.* The data for these objects are listed in Table V, which calls but for a few explanations.

Plaskett adopts for all his forty-five stars $M = -4{\cdot}0$ as the average absolute magnitude. But on further scrutiny of his paper I find that for his 'groups' I, II, and III of the O-stars proper, consisting of the stars labelled here by 1 to 12, 13 to 26, and 27 to 35,

$$\overline{M} = -4{\cdot}90, \quad -3{\cdot}88, \text{ and } -0{\cdot}45$$

respectively. Therefore, m being the observed apparent magnitude, I have calculated the distances of these O-stars by the formulae

$$\log(r) - 1 = \tfrac{1}{5}(m + 4{\cdot}90), \quad \tfrac{1}{5}(m + 3{\cdot}88), \quad \text{and } \tfrac{1}{5}(m + 0{\cdot}45),$$

where (r) stands for the number of *parsecs* contained in the distance. Further, since all the distances implied are comparatively small (below 1,500 parsecs), we have $v_t = \mathfrak{R} \sin\sigma . \mu \doteq r\mu$, and since parsec/year $= 3{\cdot}26$ light-years/year $= 3{\cdot}26 . 3 . 10^5$ km./sec., we have

$$v_t = 4{\cdot}74(r)(\mu),$$

where (μ) is the number of seconds of arc per annum contained in μ. Finally, the resultant velocity is given by $v^2 = v_r^2 + v_t^2$.

The distances, then, in Table V are in parsecs and all the velocities in km./sec. Merely for convenience in applying the two-groups method eighteen O-stars are inscribed in one, and the remaining seventeen in another section. They are arranged in the order of increasing r.

The means for all thirty-five stars are $\overline{r} = 819$, $\overline{v} = 63$, and, putting $\xi = r - \overline{r}$, $\eta = v - \overline{v}$,

$$\Sigma\xi^2 = 4{\cdot}252 . 10^6, \quad \Sigma\eta^2 = 4{\cdot}894 . 10^4, \quad \Sigma\xi\eta = +137{,}000;$$

whence the correlation coefficient between the resultant velocity and distance,

$$k(v,r) = +0{\cdot}300 \pm 0{\cdot}104, \quad . \quad . \quad . \quad . \quad (a)$$

that is, just about *three* times its P.E. This is a rather weak correlation; yet, in view of the small r's not to be despised.

* A curious property of all forty-five stars is exhibited in Note 4 *a*.

TABLE V. *O-stars*

No.	r	v_t	v_r	v	No.	r	v_t	v_r	v
1.	311	18	13	22	19.	912	68	12	69
2.	315	41	11	42½	20.	916	43	8	44
3.	330	64	42	77	21.	951	18½	6	19
4.	339	14½	17	22	22.	982	77	4	77
5.	355	71	43	83	23.	1000	54	24	59
6.	358	2	21½	22	24.	1033	54	11	55
7.	372	30	24	38	25.	1040	123	63	138
8.	419	109	39	116	26.	1040	141	31	144
9.	427	42	28	50	27.	1042	64	74	98
10.	427	20	37	42	28.	1080	29	18	34
11.	515	27	36	45	29.	1120	27	16	31
12.	617	41½	65	77	30.	1127	27	24	36
13.	718	38·8	6	39	31.	1310	149	9	149
14.	724	39	46	60	32.	1320	83½	24	87
15.	801	11	1	11½	33.	1360	101	3	101
16.	824	28	34	44	34.	1370	62	36	72
17.	866	121	59	135	35.	1440	55	8	55½
18.	887	21	15	26					

Let us now apply to these thirty-five O-stars divided into two groups, one of eighteen and the other of seventeen stars, the statistical formula (e) of Note 3. We have, by Table V,

$$\overline{\left(\frac{r v_r}{v}\right)^2_1} = 1{\cdot}016\,.\,10^5 \qquad \overline{v_1^2} = 3794$$

$$\overline{\left(\frac{r v_r}{v}\right)^2_2} = 1{\cdot}861\,.\,10^5 \qquad \overline{v_2^2} = 6854$$

$$\overline{\Delta} = 8{\cdot}45\,.\,10^4 \qquad \overline{\Delta} = 3060\,.$$

Thus,

$$\left.\begin{aligned}\mathfrak{R} &= 1{\cdot}58\,.\,10^6 \text{ parsec}\\ &= 3{\cdot}25\,.\,10^{11} \text{ a.u.}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (b)$$

which is remarkably close to the value (f), Note 3, deduced from the Cepheid variables. Both values are, roughly, twenty times smaller than that derived from the globular clusters. But in view of this coincidence (3·01 and 3·25.10¹¹), and the more reliable

distance estimates of both these groups of stars, as compared with the clusters and the Magellanic Clouds, one cannot help feeling inclined to rely more upon the curvature radius derived from these much nearer luminaries.

Of course, several more such groups of stars are greatly desirable before one forms a final judgement about this important universal constant. For the present, however, none are available or, at least, accessible to the author. [This was written Dec. 25, 1928. Some months later a memoir by Young and Harper had furnished very abundant material, again corroborating the last value of the radius. See Note 8.]

Before leaving these, at any rate, promising O-stars one more remark.

In his contribution to the *Seeliger-Festschrift* (1924), p. 331, Prof. Plaskett says that the last *nine* stars of Table V, 27 to 35, are 'of poorly determined proper motions'. It may therefore be worth our while to recalculate $\mathfrak{R}$ from the first twenty-six stars of that table. If we again apply the formula (e) of Note 3, dividing these stars into two groups of thirteen each, the result is

$$\left.\begin{aligned}\mathfrak{R} &= 1{\cdot}38.10^{6}\text{ parsec}\\ &= 2{\cdot}85.10^{11}\text{ a.u.}\end{aligned}\right\} \qquad (c)$$

This is only by 12 per cent. smaller than the last value, and somewhat nearer the Cepheid-value obtained in the preceding Note.

If the $\mathfrak{R}$-value derived from the O-stars and the Cepheids were supported by other evidence and a radius of the order 1·5 million parsecs ultimately accepted, the natural time unit, our *cosmic day*, would be of the order of 30 million years only.

4a. Correlation between Resultant Velocity and Distance for a Group of 35 O-Stars and 10 B-Stars.

Adopting Prof. Plaskett's average absolute magnitude $M = -4{\cdot}0$ for all the forty-five stars tabulated in his paper (cf. preceding note), although ten of these are avowedly B-stars, one finds the distances r (parsecs) and resultant velocities v (km./sec.), whose excesses over the general means $\bar{r} = 1222$, $\bar{v} = 137{\cdot}0$,

$$\xi = r - \bar{r}, \quad \eta = v - \bar{v},$$

are collected in Table VI, where the first thirty-five items belong to the O- and the last ten to the B-stars. The first two columns contain ξ and η, and the third only *the sign* of $\xi\eta$.

TABLE VI. *O-stars and B-stars*

ξ	η	$(\xi\eta)$	ξ	η	$(\xi\eta)$
(1) −815	−67	+	(24) −82	−101	+
−748	−111	+	−217	−117	+
−882	−97	+	−375	−126	+
−477	−107	+	+968	−27	−
−985	−115	+	+388	+77	+
−678	−98	+	+688	+21	+
−619	−91	+	+928	+424	+
−742	−84	+	+468	+194	+
−573	−86	+	+598	+232	+
−540	−99	+	+968	+79	+
−533	−52	+	+518	−61	−
−617	−108	+	(35) +368	−19	−
+158	+21	+			
−306	+4	−	(36) +738	+222	+
+228	−63	−	+388	+171	+
−160	−75	+	+428	−19	−
−125	+15	−	+158	+75	+
−125	+8	−	−142	+11	−
+288	−79	−	+378	+335	+
−284	−98	+	−318	−98	+
+218	−30	−	−293	−73	+
−92	−106	+	+598	+84	+
(23) +178	−46	−	(45) +1068	+292	+

Now, the remarkable property of these forty-five pairs of data r, v is that they show a very strong correlation. A mere glance on the $(\xi\eta)$ column shows this, thirty-three signs being positive and only twelve negative. And, in fact, calculating the numbers required for the Bravais-Pearson correlation coefficient (k) and its P.E. (Q), one finds

$$\Sigma\xi^2 = 1{\cdot}366.10^7, \quad \Sigma\eta^2 = 7{\cdot}797.10^5, \quad \Sigma\xi\eta = +2{\cdot}298.10^7,$$

whence

$$k = +0{\cdot}704, \quad \text{and} \quad Q = \frac{0{\cdot}675\,(1-k^2)}{\sqrt{45}} = 0{\cdot}051,$$

so that $k/Q = 13{\cdot}9$. In fine, the correlation coefficient between v and r is *fourteen* times its probable error, which testifies for a very strong correlation indeed. Unfortunately, however, this correlation offers no evidence for the isotropic cosmos, that is to say, for the rôle of the term $r/\mathfrak{R}$ in our Doppler-effect formula. For all the large resultant velocities v at large distances r are mainly due to the large transversal velocities v_t and these do not (apart from second-order terms) depend on $\mathfrak{R}$, and thus offer no testimony for a finite curvature radius. This is my reason for relegating Table VI to a separate Note, the property just exhibited having seemed a sufficient justification for submitting this set of data to the reader's attention at all. It is just a curious feature of Prof. Plaskett's forty-five stars. It may be reducible to the circumstance that by ascribing to all these stars $M = -4{\cdot}0$, the luminosity of some of them has been over-estimated, thus giving large r's, and thence large $v_t = r\mu$.

5. Secular Motion of Perihelion of a Planet Revolving about a Fixed Mass-Centre in Isotropic Spacetime.

The differential equation of the planetary orbit is, by (105′), with $u = 1/\mathfrak{R}\sin\sigma$,

$$\frac{d^2u}{d\theta^2} + u = \frac{L}{p^2} + 3Lu^2 - \frac{1}{p^2\mathfrak{R}^2u^3}. \quad . \quad . \quad . \quad (a)$$

To satisfy this approximately put

$$u = \frac{L}{p^2}[1 + \epsilon\cos(\theta - \varpi)],$$

where ϵ (eccentricity) is constant and ϖ is to be considered as slowly variable (slowly moving perihelion). Then, neglecting

$$\left(\frac{d\varpi}{d\theta}\right)^2 \text{ and } \frac{d^2\varpi}{d\theta^2},$$

$$\frac{d\varpi}{d\theta} = \frac{3L^2}{2p^2}[1 + \epsilon\cos(\theta - \varpi)]^2 - \frac{p^6}{2\mathfrak{R}^2L^4}[1 + \epsilon\cos(\theta - \varpi)]^{-3},$$

whence, integrating over $x = \theta - \varpi$ from 0 to 2π, the motion of the perihelion per period of revolution,

$$\delta\varpi = \frac{6\pi L^2}{p^2} - \frac{p^6}{2\mathfrak{R}^2 L^4}\int_0^{2\pi} \frac{(1+\epsilon\cos x)^{-3}}{\epsilon\cos x}\,dx,$$

and, for small eccentricity, up to ϵ^4,

$$\delta\varpi = \frac{6\pi L^2}{p^2} + \frac{3\pi p^6}{\mathfrak{R}^2 L^4}\left(1+\frac{5}{3}\epsilon^2\right).$$

The first term represents the well-known Einsteinian perihelion motion (43″ per century for Mercury), corresponding to $\mathfrak{R} = \infty$, say $\delta\varpi_0$,

$$\delta\varpi_0 = \frac{6\pi L^2}{p^2} = \frac{24\pi^3 a^2}{c^2 T^2 (1-\epsilon^2)},$$

if a be the semi-major axis of the orbit and T the period of revolution. Thus our result may be written

$$\frac{\delta\varpi - \delta\varpi_0}{\delta\varpi_0} = \frac{3\pi}{4}\left(1-\frac{4}{3}\epsilon^2\right)\left(\frac{cT}{\pi\mathfrak{R}}\right)^2. \quad . \quad . \quad . \quad . \qquad (b)$$

The effect of a finite curvature radius upon the secular motion of the perihelion of a planet is, apart from the moderate numerical factor, of the order of a squared 'light-year' (the planets' light-year, that is) divided by the square of the total length of a straight line in elliptic space, and is thus hopelessly small to be ever detected. In fact, even if $\mathfrak{R}$ were only of the order of a million parsecs or 3·26 million (terrestrial) light-years, the squared factor in (b) would, for our Earth, be of the order 10^{-14} and $\delta\varpi - \delta\varpi_0$ of the order of $2''.10^{-15}$, and even for a planet with a period amounting to a hundred years, the perihelion-effect of a finite curvature radius would still be as small as $8''.10^{-9}$ per century.

Likewise the effects upon the remaining elements of Keplerian motion can be shown to be practically evanescent.

6. Fluid Sphere in Equilibrium in Isotropic Space-time.

For small $r/\mathfrak{R}$ this problem can be treated with sufficient accuracy by superposing upon the Newtonian gravitation the

centrifugal acceleration $\omega^2\mathbf{r} = c^2\mathbf{r}/\mathfrak{R}^2$, where $\mathbf{r}$ is the ordinary vector drawn from the origin O to any point.

Thus, if p and ρ be the pressure and the density of the fluid, M the total mass of the fluid globe, and a its radius,

$$\frac{1}{\rho}\nabla p = -\frac{M}{a^3}\mathbf{r} + \frac{c^2}{\mathfrak{R}^2}\mathbf{r}$$

or, with $L = M/c^2$,

$$\frac{1}{\rho}\nabla p = \frac{c^2}{\mathfrak{R}^2}\left[1 - \frac{L\mathfrak{R}^2}{a^3}\right]\mathbf{r} = \frac{c^2}{\mathfrak{R}^2}\left[1 - \left(\frac{a^*}{a}\right)^3\right]\mathbf{r}, \quad . \quad . \quad (a)$$

where $a^* = (L\mathfrak{R}^2)^{\frac{1}{3}}$ is the 'critical radius' of a mass-centre M. At the surface of the sphere, $r = a$, the condition is $p = 0$.

The density may be constant or variable (a function of r). Since ρ is positive, it follows from (a) that p would *increase* with r if $a > a^*$, and thus, since $p_a = 0$, the pressure would have to be negative throughout the sphere. Thus, if we require $p \geqslant 0$, *the upper limit* for the radius of a fluid globe in equilibrium is

$$a^* = (L\mathfrak{R}^2)^{\frac{1}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

In fine, the critical radius a^* plays here the same rôle as for a point-mass M. Notice that $a = a^*$ would mean $p = 0$ throughout, or just the breaking-point of the globe. Thus the necessary condition for a properly permanent globe is $a < a^*$ (if it has no spin in the ordinary sense of the word; and the more so, if it is spinning, as most nebulae are). The critical density is again $\rho^* = Lc^2 \Big/ \frac{4\pi}{3} a^{*3} = 3c^2/4\pi\mathfrak{R}$, as on p. 174.

Notice, further, that this property is entirely independent of the nature of the fluid (gas or liquid), that is to say, of the form of the relation $f(p, \rho) = 0$. If this relation be given, p and ρ can be found as functions of r from (a).

7. Illuminated Spacetime: Effects of Isotropic Radiation spread over Elliptic Space.

The isotropic world with the elliptic space as its section has, throughout the book, been assumed to be *empty*, apart from the 'test particle' (planet or star), that is to say devoid

of matter in its usual sense as well as of energy, say electromagnetic or more especially radiant energy. In other words, we have, in Einstein's field-equations, equated to zero the whole tensor of matter or energy-tensor $T_{\iota\kappa}$.

Now, although stray particles of matter (proper), as molecules, atoms, protons, or electrons, are rather scarce in interstellar, and still more so in intergalactic regions,* *radiant energy* or, more plainly, light visible and invisible, emitted from stars and nebulae, so plentifully, can without exaggeration be said to be omnipresent, spread all over the space. Nay, its total mass-equivalent (E/c^2) amounts to a surprisingly high figure. To realize this it is enough to recall that our sun alone radiates, in its present state, at least $3{\cdot}8 . 10^{33}$ ergs per second, or $\frac{4}{3} . 10^{20}$ grams per year, and since it is pretty certain that it did this, and even more copiously, at least during the last thousand million years, and since but very little of this radiation has been intercepted, it is certain that the mass of solar radiation now abroad in space is not less than $1{\cdot}33 . 10^{29}$ grams, which, though only $6{\cdot}7 . 10^{-5}$ suns (mass unit), is an imposing mass. Our galaxy consists of, say, $3 . 10^9$ stars, of which many are more lavish than the sun. Thus the radiant energy originally emitted during that time by all these stars and now abroad partly within and partly outside, in intergalactic regions, has the prodigious mass of 200,000 suns. And then there are certainly a good many millions of such galaxies. It is true that all this 'radiant mass' (to have a brief name for E/c^2) is distributed over a huge volume, viz. $V = \pi^2 \mathfrak{R}^3$, so that its mean density does not amount to very much.† Yet it would be unwise just to neglect it.

It has therefore seemed worth while to take account of this stray light, so to speak. In other words, while hitherto we have

* See, for instance, Eddington's *Internal Constitution of the Stars*, Cambridge, 1926.

† According to Eddington's quotation (loc. cit.), the total density of radiation 'received *by us* from the stars' is $7{\cdot}7 . 10^{-13}$ ergs/cm.^{3}, whence the corresponding mass-density, $\rho = 8{\cdot}6 . 10^{-34}$ gr./cm.^{3}. The italicized words imply, of course, that this density holds in inter-stellar regions within our galaxy. Somewhere half-way between the Milky Way and the nebula in Andromeda, say, ρ may be a good deal smaller.

considered an empty, *dark* spacetime or world, we now propose to investigate an *illuminated* world. And as this illumination is actually provided by myriads of celestial bodies distributed more or less haphazardly, it is reasonable to treat it as *isotropic radiation.* Now, this has the capital property that the light pressure, a hydrostatic pressure, associated with it is just *one-third* of the energy density, say,

$$p = \tfrac{1}{3}\rho. \qquad \text{(a)}$$

The corresponding energy-tensor is therefore, in orthogonal coordinates,

$$T_{ii} = -\tfrac{1}{3}\rho g_{ii}, \quad T_{44} = \rho g_{44}, \quad i = 1, 2, 3, \qquad \text{(b)}$$

and has the remarkable property that its scalar, $T = \rho - 3p$, vanishes. (This, by the way, is also the property of every electromagnetic energy-tensor; cf. *Theory of Relativity*, p. 449.) Thus, Einstein's cosmologically amplified field-equations

$$R_{\iota\kappa} - \tfrac{1}{2}(R - 2\lambda)\, g_{\iota\kappa} = -\kappa T_{\iota\kappa},$$

which give at once $\lambda = R\tfrac{1}{4} = \tfrac{1}{4} g^{\iota\kappa} R_{\iota\kappa}$, become

$$R_{ii} = \left(\tfrac{1}{4} R + \frac{\kappa\rho}{3}\right) g_{ii}, \quad R_{44} = (\tfrac{1}{4} R - \kappa\rho)\, g_{44}. \qquad \text{(c)}$$

Let us try to solve these equations by the already familiar radially symmetrical form of the line-element,

$$ds^2 = g_1 dx^2 - x^2(d\phi^2 + \sin^2\phi\, d\theta^2) + g_4 c^2 dt^2, \qquad \text{(d)}$$

assuming, that is, ρ as function of x alone, and thus a *globe*, of any radius, filled with isotropic radiation. (Its radius, if we like, may be made equal $\frac{\pi}{2}\mathfrak{R}$, when the whole elliptic space will be full of radiation, as very likely it is.)

The problem consists in determining g_1, g_4, and ρ as functions of x.

Now, in familiar symbols, the second of (c) is

$$-R_{22} = 1 + \frac{1}{g_1}\left[1 + \frac{x}{2}(h_4' - h_1')\right] = -\left(\frac{R}{4} + \frac{\kappa\rho}{3}\right) x^2, \qquad (e_1)$$

the third says the same thing (since $R_{33} = R_{22} \,.\, \sin^2\phi$), the fourth is

$$R_{44} = \frac{g_4}{g_1}\left[R_{11} + \frac{1}{x}(h_1' + h_4')\right] = (\tfrac{1}{4} R - \kappa\rho)\, g_4$$

or, replacing R_{11} by its value given in (c),

$$h_1' + h_4' = -\tfrac{4}{3}\,\kappa\rho\,x g_1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e_2)$$

Instead of the first of (c) written out explicitly, it is more convenient to apply the first of the so-called equations of matter,* which are only a consequence of the gravitational field-equations, and this gives without trouble, $\rho\sqrt{g_4}$ = constant. Since, without any loss to generality we can put $g_4(0) = 1$, we have

$$\rho = \rho_0/\sqrt{g_4}, \quad . \quad . \quad . \quad . \quad . \quad .. \qquad (e_3)$$

where ρ_0 is the value of ρ (*system* density of radiant mass) at the origin O of the coordinates. It remains to substitute this expression into the differential equations (e_1), (e_2), and to solve them for g_1, g_4 as functions of x.

Subtracting (e_2) from (e_1), we have

$$\frac{d}{dx}\left(1+\frac{1}{g_1}\right) + \frac{1}{x}\left(1+\frac{1}{g_1}\right) = (\tfrac{1}{4}\,R + \kappa\rho)\,x,$$

whence, in absence of a singularity (mass-centre) at O,

$$-\frac{1}{g_1} = 1 - x^2/\mathfrak{R}^2, \quad \mathfrak{R}^2 = \frac{3}{\lambda+\kappa\rho}. \quad . \quad . \quad . \quad . \qquad (f)$$

It remains to find g_4 from (e_2). Now, this can be written

$$\frac{d\sqrt{g_4}}{dx} + \left(\frac{d}{dx}\log\sqrt{-g_1}\right)\cdot\sqrt{g_4} = -\tfrac{2}{3}\,\kappa\rho_0\,x g_1,$$

which is Euler's linear equation. Its complete solution is

$$\sqrt{g_4} = e^{-\int d\log\sqrt{-g_1}}\left\{C - \tfrac{2}{3}\,\kappa\rho_0\int x g_1\, e^{\int d\log\sqrt{-g_1}}\,dx\right\},$$

where C is an arbitrary constant, or

$$\sqrt{g_4} = \frac{1}{\sqrt{-g_1}}\left\{C - \frac{2i}{3}\,\kappa\rho_0\int x g_1^{3/2}\,dx\right\}, \quad i = \sqrt{-1}, \ . \qquad (g)$$

where g_1 is given by (f), and ρ by (e_3). Thus $\mathfrak{R}$, containing ρ and $\lambda = \frac{1}{4}\,R = g^{\iota\kappa}R_{\iota\kappa}/4$, might depend on g_4 and directly on x, and the evaluation of $\sqrt{g_4}$ from (g) would be a highly complicated affair.

* Cf. *Theory of Relativity*, p. 424.

Fortunately, however, the combination $\lambda+\kappa\rho$ is a constant. In fact, we may write

$$R = \frac{1}{g_1}R_{11}+\frac{2}{g^2}R_{22}+\frac{1}{g_4}R_{44} = \frac{2}{g_1}R_{11}-\frac{2}{x^2}R_{22}+\frac{h_1'+h_4'}{xg_1},$$

$$\lambda = \tfrac{1}{4}R = \frac{1}{x^2 g_1}\left[1+\frac{x}{2}(h_4'-h_1')\right]+\frac{1}{x^2}+\frac{h_1'+h_4'}{2xg_1}+\tfrac{1}{4}\kappa\rho$$

or, after simple reductions,

$$\lambda = \kappa\rho+\tfrac{3}{2}\frac{g_4'}{xg_1g_4},$$

and, eliminating g_4' with the aid of (e_2),

$$\lambda = -\kappa\rho+\frac{3}{2x}\frac{d}{dx}\left(\frac{1}{g_1}\right) = -\kappa\rho+\frac{3}{\Re^2}\left[1-\frac{x}{\Re}\frac{d\Re}{dx}\right],$$

the latter by (f). Thus,

$$\frac{1}{\Re^2} = \frac{\lambda+\kappa\rho}{3} = \frac{1}{\Re^2}\left[1-\frac{x}{\Re}\frac{d\Re}{dx}\right],$$

from which we see two things: first, that $d\Re/dx = o$, i.e. that $\Re$ is a *constant* throughout the space, and second, that *its numerical value remains free*, not determined by the density ρ or ρ_0 of the radiant energy. The value of $\Re$ is to be found, independently, through observations (viz. the Doppler effect).

Under these circumstances the evaluation of the integral in formula (g) becomes an easy matter. In fact put, as on previous occasions, $x = \Re\sin\sigma$, $\sigma = r/\Re$. Then, by (f),

$$g_1 = -\sec^2\sigma, \quad \frac{1}{\sqrt{g_1}} = i\cos\sigma,$$

$$\int xg_1^{3/2}dx = i\Re^2\int\frac{\sin\sigma}{\cos^2\sigma}d\sigma = i\Re^2\sec\sigma,$$

and ultimately, by (g),

$$\sqrt{g_4} = C\,.\,\cos\sigma+\tfrac{2}{3}\kappa\rho_0\Re^2.$$

Now, without any loss to generality we can put $g_4(o) = 1$. Thus $C = 1-\frac{2}{3}\kappa\rho_0\Re^2$, and

$$\sqrt{g_4} = \cos\sigma+\tfrac{2}{3}\kappa\rho_0\Re^2(1-\cos\sigma). \quad . \quad . \quad . \quad . \quad (h)$$

Since $dx = \cos\sigma\, dr$, and therefore $g_1 dx^2 = -dr^2$, the required solution, of form (*d*), ultimately becomes

$$ds^2 = g_4 c^2 dt^2 - dl^2, \quad \ldots \ldots \ldots \quad (i)$$

where g_4 is as in (*h*) and dl^2 is the familiar line-element of an elliptic three-space of curvature radius $\mathfrak{R}$. As we just saw, this radius is, as in empty space, constant, while its value does not at all depend on the intensity of illumination.

Since $\kappa = 8\pi k/c^2$, where k is the gravitation constant, we have

$$\tfrac{2}{3}\kappa\rho_0 = \frac{16\pi}{3} \frac{k\rho_0}{c^2},$$

which has the dimensions of a reciprocal area. Thus, if we introduce the length λ_0 through $k\rho_0/c^2 = 1/\lambda_0^2$,

$$\sqrt{g_4} = \cos\sigma + \frac{16\pi}{3}\left(\frac{\mathfrak{R}}{\lambda_0}\right)^2 (1-\cos\sigma). \quad \ldots \quad (h')$$

For $\rho_0 = o$ ($\lambda_0 = \infty$) our solution reduces to that found by de Sitter, $g_4 = \cos^2\sigma$, as it should.

With regard to the density of distribution of the radiant energy it is important to notice that ρ, for which we have found $\rho_0/\sqrt{g_4}$, is the system density. The *natural measure* of density can readily be shown to be $\rho\sqrt{g_4}$ and is therefore *constant*, namely ρ_0 itself.

As to the physical consequences of the solution just obtained, it will be enough to consider here the Doppler effect, for a star and an observer in free (inertial) relative motion, as influenced by the illumination.

The Doppler effect is, again,

$$D = \frac{\delta\lambda}{\lambda} = \frac{ds}{ds'} - 1,$$

and if the origin of coordinates is placed in the star,

$$cdt - ds' = dr/\sqrt{g_4}.$$

Further, since the observing station (the sun) is supposed to move inertially,

$$g_4 c dt/ds = k = \text{const.}$$

Thus the formula for the Doppler effect becomes, rigorously,

$$D+1 = \frac{g_4}{k\left[1 - \frac{1}{\sqrt{g_4}} \frac{dr}{c\,dt}\right]}.$$

Here, as follows readily from the developed form of $\delta \int ds = 0$,

$$\frac{1}{c}\frac{dr}{dt} = \sqrt{g_4}\sqrt{1 - \frac{g_4}{k^2} - \frac{p^2 g_4}{k^2 \Re^2 \sin^2\sigma}},$$

where $p = \Re^2 \sin^2\sigma\, d\theta/ds =$ const. is a first integral of the equations of inertial motion.

For small values of $\sigma = r/\Re$, that is to say, neglecting σ^4 in presence of unity, also $\sigma^2 v_r^2/c^2$, our formula, after simple reductions, becomes

$$D^2 = \frac{v_r^2}{c^2} = \left(1 - \frac{r_0^2}{r^2}\right)(\beta_0^2 + N\sigma^2), \quad . \quad . \quad . \quad . \quad (j)$$

where
$$N = 1 - \frac{16\pi}{3}\frac{\Re^2}{\lambda_0^2},$$

and σ_0, $\beta_0 = v_0/c$ refer to the perihelion of the star's orbit.

The only difference, as compared with dark spacetime, is that σ^2 in the second factor is replaced by $N\sigma^2$, i.e. $r^2/\Re^2$ by $r^2 N/\Re^2$. In fine, $\Re$, as evaluated from the velocities of celestial objects of known distance, is replaced by

$$\Re' = \Re/\sqrt{N} = \Re/\sqrt{1 - 16\pi\Re^2/3\lambda_0^2}. \quad . \quad . \quad . \quad (k)$$

Thus, what we have determined, say from the Cepheids and the O-stars, is $\Re'$. Having found this, we can determine $\Re$ by the last formula. While $\Re'$ is of the order of 10^{11} to 10^{12} astronomical units, λ_0 is some 10^4 or 10^5 times greater. In fact, according to Eddington's quotation (*l. c.*) the density of radiation in interstellar regions is $7{\cdot}7 \,.\, 10^{-13}$ erg/cm³, whence $k\rho_0 = 5{\cdot}73 \,.\, 10^{-41}$ astronomical mass units per cm³, and

$$\lambda_0 = \frac{c}{\sqrt{k\rho_0}} = 2{\cdot}65 \,.\, 10^{17} \text{ a.u.}$$

Possibly, if ultra-violet and X-rays were taken into account, ρ_0 would be several times greater, and λ_0 might drop to 10^{16} or even

10^{15} a. u. At any rate, however, the reduction formula (k) can safely be replaced by

$$\Re = \Re' \left\{1 - \frac{8\pi}{3}\left(\frac{\Re'}{\lambda_0}\right)^2\right\}.$$

8. Determination of the Curvature Radius based on 459 Stars from Young and Harper's list.

The values of the curvature radius of spacetime obtained, in Notes 3 and 4, from the twenty-four Cepheids and the thirty-five O-stars have found recently (July 1929) an excellent confirmation and a stout support in the result yielded by a much greater batch of other stars. The data for these objects were taken from R. K. Young and W. E. Harper's valuable memoir on *The absolute Magnitudes and Parallaxes of 1105 Stars* (Publ. Dominion Astrophys. Observatory, Victoria, B.C., vol. III, No. 1, Ottawa, 1924), supplemented by a list of radial velocities v_r relative to the sun (not corrected for 'the solar motion') kindly communicated to me by Prof. Young. Having taken into account only those objects for which $r \geqslant 50$ parsecs and $v < 100$ km/sec., I have thus obtained as many as 459 stars for which all the required data, r, v_t, and v_r (whence also $v^2 = v_r^2 + v_t^2$), are known. The complete material, ordered according to increasing distance, is given in the following table. The numbers in the first column are (with a few exceptions) those of Boss's catalogue. The distances are in parsecs, and the velocities in km/sec.

459 Stars from Young and Harper's List.

Object	r	μ	v_t	v_r
2	50	0·038	9	3
441	50	0·093	22	1
1222	50	0·170	40	30
1974	50	0·012	3	32
3008	50	0·142	34	16
3681	50	0·151	36	7
3717	50	0·148	35	14
4101	50	0·041	9·7	9·7
4411	50	0·077	18	10
4811	50	0·141	33	12

Object	r	μ	v_t	v_r
4863	50	0·079	18	28
4986	50	0·010	2·4	23
5875	50	0·064	15	3·8
206	53	0·060	15	12
304	53	0·043	11	10
393	53	0·086	22	14
654	53	0·102	26	1
1042	53	0·021	5	18
1169	53	0·098	25	2
2302	53	0·024	6	27
2632	53	0·329	83	27
3090	53	0·134	34	9
4609	53	0·275	69	17
3741	53	0·050	13	25
5075	53	0·072	18	20
5027	53	0·039	10	22
5184	53	0·158	40	17
5790	53	0·328	82	55
6040	53	0·064	16	14
6148	53	0·020	5	11
271	56	0·084	22	30
281	56	0·050	13	5
335	56	0·034	9	15
1429	56	0·034	9	19
1987	56	0·119	32	20
1055	56	0·113	30	27
4437	56	0·036	9·6	32
5031	56	0·144	38	21
5452	56	0·061	16	17
5523	56	0·113	30	19
5800	56	0·143	38	0·2
973	59	0·054	15	38
1627	59	0·331	93	36
1720	59	0·051	14	19
2528	59	0·076	21	9
3447	59	0·123	34	9·5
3668	59	0·120	34	3·5
3960	59	0·057	16	48
4057	59	0·087	24	13
4797	59	0·004	1·1	23

Object	*r*	μ	v_t	v_r
4924	59	0·017	4·8	0·7
5089	59	0·129	36	42
5294	59	0·030	8·4	7
5401	59	0·128	36	22
5724	59	0·051	14	7
6094	59	0·083	23	5
179	62·5	0·032	9·5	5
226	62·5	0·082	24	8
497br.	62·5	0·088	26	19
497ft.	62·5	0·088	26	20
847	62·5	0·007	2	12
1122	62·5	0·068	20	38
1166	62·5	0·032	9·5	12
1565	62·5	0·273	81	20
1604	62·5	0·130	39	55
1607	62·5	0·021	6	14
1753	62·5	0·047	14	14
1950	62·5	0·044	13	18
1981	62·5	0·031	9	5
2229	62·5	0·243	72	37
3142	62·5	0·040	12	98
3171	62·5	0·163	49	5
3470	62·5	0·130	39	9·6
3673	62·5	0·013	4	26
3497	62·5	0·134	39	14
4782	62·5	0·082	25	3
5373	62·5	0·102	31	1·5
5504	62·5	0·042	12	7
5852	62·5	0·100	30	9
5924	62·5	0·108	32	14
6037	62·5	0·136	41	6
704	67	0·072	23	15
1043	67	0·112	36	50
1061	67	0·007	2	21
1129	67	0·116	37	23
1219	67	0·068	22	5
2130	67	0·106	34	71
2195	67	0·072	23	23
2251	67	0·075	24	11
2348	67	0·051	16	16

Object	r	μ	v_t	v_r
2586	67	0·116	37	20
3753	67	0·007	2·2	22
3572	67	0·104	33	5
3795	67	0·228	72	67
4522	67	0·050	16	33
5045	67	0·079	25	24
5584	67	0·028	9	5
6024	67	0·195	62	9
234	71	0·042	14	6
270	71	0·022	7	16
307	71	0·078	26	14
792	71	0·114	38	51
835	71	0·033	11	21
1040	71	0·120	40	36
1367	71	0·139	47	20
2001	71	0·123	41	7
2008	71	0·106	36	4
2602	71	0·147	49	27
2656	71	0·078	26	1
3089	71	0·185	62	51
3141	71	0·133	45	28
4003	71	0·040	13	4
4075	71	0·153	52	17
4240	71	0·084	28	19
4244	71	0·120	40	46
4015	71	0·108	36	38
4186	71	0·028	9·4	8·4
4364	71	0·050	17	56
4594	71	0·043	14	15
4790	71	0·089	30	20
R.Y. Boötis	71	0·051	17	1
3555	71	0·067	23	10
5023	71	0·038	13	2·3
5204	71	0·050	17	68
5230	71	0·048	16	23
5235	71	0·044	15	13
5732	71	0·052	18	10·5
5940	71	0·234	79	9
200	77	0·099	36	47
248	77	0·200	73	10

Object	r	μ	v_t	v_r
325	77	0·042	15	15
338	77	0·038	14	6
1350	77	0·030	11	12
1445	77	0·036	13	45
1728	77	0·009	3	10
2620	77	0·153	56	44
2910	77	0·112	47	23
4593	77	0·211	77	7
4626	77	0·060	22	9
5716	77	0·031	11	14
6112	77	0·060	22	9
6015	77	0·033	12	4·4
5660	77	0·089	32	34·5
6001	77	0·059	22	11
3856	77	0·047	17	15
3907	77	0·122	45	10·5
4351	77	0·090	33	12
49	83	0·076	30	10
53	83	0·040	16	19
368	83	0·142	56	2
378	83	0·018	7	1
439	83	0·040	16	0·1
545	83	0·032	13	3
755	83	0·029	11	1
770	83	0·048	19	4
840	83	0·141	55	5
1004	83	0·123	48	37
1117	83	0·068	27	4
1191	83	0·068	27	8
1552	83	0·047	18	13
1559	83	0·030	12	23
1650	83	0·043	17	3
1694	83	0·061	24	17
1822	83	0·086	34	3
1980	83	0·044	17	1
2234	83	0·041	16	14
2238	83	0·094	37	16
2392	83	0·117	46	59
2550	83	0·053	21	28
2980	83	0·089	35	1

Object	r	μ	v_t	v_r
3027	83	0·148	58	2
3149	83	0·167	66	8
3181	83	0·031	12	26
3559	83	0·114	45	11
3278	83	0·039	15	21
3940	83	0·186	73	21
3992	83	0·036	14	3
4031	83	0·045	18	60
4651	83	0·069	27	19
4193	83	0·032	13	3
4423	83	0·100	39	0·4
5021	83	0·034	13	4
5137	83	0·022	9	0·3
5674	83	0·146	57	16
5522	83	0·021	8	18
5709	83	0·027	11	7
330	91	0·047	20	11
672	91	0·035	15	5
952	91	0·117	50	22
2040	91	0·094	41	78
2205	91	0·091	39	19
3030	91	0·076	33	25
H.R. 6807	91	0·012	5	26
3230	91	0·014	6	12
3401	91	0·088	38	15
3649	91	0·038	16	10
3816	91	0·070	30	21
4130	91	0·049	21	3
4393	91	0·048	21	13
4547	91	0·011	5	23
5187	91	0·006	3	3
5201	91	0·030	13	12
5590	91	0·033	14	24
5746	91	0·024	10	7
5838	91	0·173	75	6
5990	91	0·130	56	39
31	100	0·097	46	45
54	100	0·015	7	19
134	100	0·049	23	18
173	100	0·097	46	33

Object	r	μ	v_t	v_r
402	100	0·116	55	36
555	100	0·046	22	40
670	100	0·061	29	36
775	100	0·061	29	12
795	100	0·021	10	17
1439	100	0·052	25	40
1530	100	0·029	14	8
1626	100	0·012	6	33
1628	100	0·043	20	53
1717	100	0·020	9	10
1835	100	0·110	52	18
1962	100	0·019	9	15
2271	100	0·064	30	22
2311	100	0·033	16	38
2611	100	0·042	20	33
2673	100	0·051	24	14
2828	100	0·116	55	45
2978	100	0·025	12	11
3000	100	0·029	14	10
3031	100	0·043	20	7
3402	100	0·043	20	15
3492	100	0·069	33	0·0
3533	100	0·037	18	4
3557	100	0·079	37	7
3570	100	0·043	20	10·5
3803	100	0·179	85	10
3817	100	0·033	16	16
3908	100	0·134	64	53
3982	100	0·041	19	24
4195	100	0·075	36	84
4258	100	0·087	41	11
4443	100	0·013	6	20
4472	100	0·051	24	1·5
4506	100	0·024	11	25
4510	100	0·051	24	15
4535	100	0·008	4	27
4798	100	0·092	44	2
4977	100	0·053	25	41
5255	100	0·014	7	19

Object	r	μ	v_t	v_r
5676	100	0·018	8·5	7
5727	100	0·025	12	4
5388	100	0·040	19	27
5495	100	0·075	36	3
5823	100	0·141	67	5·5
5840	100	0·115	55	19
5900	100	0·032	15	13
5952	100	0·019	9	5
239	111	0·052	27	28
850	111	0·070	37	15
924	111	0·006	3	20
1468	111	0·032	17	21
1914	111	0·066	35	4
1953	111	0·062	33	47
2101	111	0·033	17	40
2232	111	0·133	70	24
2439	111	0·018	9	26
2530	111	0·134	70	38
2543	111	0·063	33	1
2660	111	0·066	35	46
3299	111	0·074	39	25
3536	111	0·025	13	17·5
3588	111	0·036	19	42
3930	111	0·113	59	46
4191	111	0·046	24	24
4307	111	0·062	33	0·3
4311	111	0·102	54	11
4471	111	0·059	31	27
4622	111	0·034	18	8·5
4756	111	0·024	13	21
4848	111	0·015	8	9
4958	111	0·049	26	25
5309	111	0·090	47	12
5751	111	0·048	25	8
5756	111	0·051	27	4·5
6097	111	0·032	17	8
96	125	0·056	33	56
191	125	0·018	10	16
367	125	0·076	45	17
499	125	0·017	10	44

Object	r	μ	v_t	v_r
1561	125	0·062	38	20
2028	125	0·032	19	3
2310	125	0·039	23	35
2411	125	0·023	14	6
2680	125	0·041	24	23
2829	125	0·010	6	7
2918	125	0·080	47	50
3007	125	0·073	43	7
3233	125	0·072	43	6
3374	125	0·033	20	1
3736	125	0·093	55	15
4606	125	0·016	9·5	1
4833	125	0·045	27	9
4912	125	0·007	4	30·5
5010	125	0·022	13	32
5049	125	0·014	8	19
5052	125	0·009	5	3
5200	125	0·008	5	14
5229	125	0·006	4	6
5431	125	0·011	6·5	19
5436	125	0·018	11	26
5567	125	0·056	33	29
6127	125	0·049	29	11
73	143	0·025	17	4
80	143	0·017	12	6
106	143	0·016	11	13
420	143	0·011	7	6
559	143	0·091	62	28
639	143	0·029	20	0·5
646	143	0·079	54	14
746	143	0·016	11	3
1557	143	0·097	66	25
1583	143	0·077	52	46
1643	143	0·018	12	14
2112	143	0·074	50	22
2338	143	0·018	12	12
2556	143	0·022	15	18
2780	143	0·043	29	25
2838	143	0·036	24	6
3157	143	0·085	58	25

Object	r	μ	v_t	v_r
3219	143	0·082	56	44
3527	143	0·036	24	14
3601	143	0·065	44	40
3859	143	0·028	19	18
3933	143	0·118	80	48
4154	143	0·008	5	8
4242	143	0·041	28	56
4336	143	0·071	48	41
4422	143	0·050	34	18
4653	143	0·057	39	14
4994	143	0·045	31	9
5527	143	0·020	14	6
5673	143	0·050	34	28
5714	143	0·014	9·5	18
5993	143	0·040	27	5
346	167	0·015	12	43
582	167	0·062	49	12
624	167	0·072	57	33
635	167	0·054	43	47
637	167	0·017	13	4
669	167	0·039	31	40
826	167	0·020	16	27
1074	167	0·026	21	10
1128	167	0·112	89	36
1187	167	0·015	12	9
1415	167	0·024	19	48
1850	167	0·054	43	13
1919	167	0·023	18	22
2308	167	0·042	33	35
2432	167	0·026	21	2
2437	167	0·040	32	18
2740	167	0·021	17	10
2895	167	0·021	17	13
2896	167	0·088	70	7
3180	167	0·059	47	25
3189	167	0·040	32	16
3346	167	0·033	26	30
3584	167	0·040	32	42
3589	167	0·005	4	10
3598	167	0·044	35	4

Object	*r*	*μ*	v_t	v_r
3764	167	0·025	20	11
4142	167	0·025	20	15
4262	167	0·055	44	20
4286	167	0·007	5·5	12
4310	167	0·032	25	0·3
4578	167	0·020	16	23
4831	167	0·016	13	15
5071	167	0·015	12	14
5280	167	0·026	21	43
5412	167	0·051	40	14
5425	167	0·053	42	4·5
6033	167	0·058	46	9
6141	167	0·072	57	1
161	200	0·064	61	26
188	200	0·089	84	5
1824	200	0·050	46	2
1868	200	0·019	18	22
1871	200	0·044	42	11
2155	200	0·022	21	29
2578	200	0·042	40	22
2671	200	0·034	32	19
2761	200	0·027	26	22
2858	200	0·045	43	3
2921	200	0·096	91	33
3083	200	0·031	29	2
3446	200	0·015	14	26
3581	200	0·070	66	42
3631	200	0·068	64	14
3761	200	0·020	19	8
4373	200	0·028	27	32
4518	200	0·050	47	68
4555	200	0·044	42	10
4629	200	0·019	18	15
4814	200	0·078	74	32
5271	200	0·041	39	66
5355	200	0·029	27	30
5459	200	0·045	43	11
5619	200	0·019	18	20
5798	200	0·034	32	45
5807	200	0·014	13	16

Object	r	μ	v_t	v_r
5872	200	0·056	53	22
5954	200	0·040	38	25
6064	200	0·050	47	5
1185	250	0·014	17	3
1479	250	0·012	14	1
1518	250	0·019	22	14
1584	250	0·015	18	41
1815	250	0·008	9	7
1846	250	0·019	22	49
2020	250	0·014	17	16
2182	250	0·038	45	29
2800	250	0·040	47	34
2824	250	0·026	31	17
2864	250	0·051	60	9
3827	250	0·080	95	8
3860	250	0·015	18	16
4329	250	0·027	32	11
4400	250	0·052	62	46
5154	250	0·059	70	70
5192	250	0·015	18	8
5205	250	0·014	17	21
5299	250	0·021	25	15
5472	250	0·027	32	19
5678	250	0·053	63	6
5797	250	0·065	77	29
5843	250	0·058	69	8
6058	250	0·037	44	2
241	333	0·023	36	27
1335	333	0·015	24	24
1629	333	0·023	36	21
2156	333	0·038	60	5
2847	333	0·028	44	14
4795	333	0·031	49	10
4817	333	0·008	13	23
4971	333	0·028	44	1
5213	333	0·020	32	14
5370	333	0·018	28	1
5416	333	0·008	13	10
5479	333	0·047	74	19
5804	333	0·021	32	12

Object	r	μ	v_t	v_r
619	500	0·006	14	4
1803	500	0·016	38	21
2953	500	0·031	73	6
3831	500	0·030	71	34
6135	500	0·004	9·5	42
1606	1000	0·014	66	10
X Cygni	1000	0·020	95	9
5098	1000	0·008	37	9
5593	1000	0·005	24	23
5931	1000	0·008	37	60

Let us apply to these data our formula (e) of Note 3, which, with $s = rv_r/v$, can be re-written

$$\Re^2 = c^2 \frac{\overline{s_2^2} - \overline{s_1^2}}{\overline{v_2^2} - \overline{v_1^2}}.$$

(Notice that s has a simple geometrical meaning, being the actual distance of a star from its perihelion.) Dividing the whole material into two groups, one of 229 and the other of 230 stars (as shown by the heavy bar above), ranging in distance from 50 to 100 and from 100 to 1,000 parsecs, I find the arithmetical means, in km²/sec² and squared parsecs,

$$\overline{v_1^2} = 1761 \qquad \overline{s_1^2} = 2113,$$

and

$$\overline{v_2^2} = 2113{\cdot}2 \qquad \overline{s_2^2} = 17102,$$

whence the curvature radius

$$\left.\begin{aligned} \Re &= 1{\cdot}96\ .\ 10^6 \text{ parsecs} \\ &= 4{\cdot}03\ .\ 10^{11} \text{ a. u.} \end{aligned}\right\}.$$

This agrees nearly enough with the values derived from the Cepheids and the O-stars. If one strikes a weighted average of the three values, the weights being made proportional to the star numbers, the result is but a little smaller, $\Re = 3{\cdot}93\ .\ 10^{11}$ a. u., and this, being based in all upon 518 stars, can be accepted with confidence as the size of the curvature radius of spacetime.

The radius yielded by 18 globular clusters and the Magellanic Clouds was about 18 times greater, but, for reasons already

indicated, it is by far less reliable. And since, according to a private letter of Prof. Shapley's, of February 1929, the distances of the clusters are being thoroughly revised and 'the Large Magellanic Cloud will probably move in considerably', it may well happen that also these more remote celestial objects will confirm the last result, a radius of the order of $4 \,.\, 10^{11}$ astronomical units.

Recalculating, with the last-obtained curvature radius $\mathfrak{R} = 3{\cdot}93 \,.\, 10^{11}$ a. u. $= 6{\cdot}27 \,.\, 10^{6}$ light-years, some of the derived magnitudes considered in the main text of the book, one finds the following values:

Cosmic day, $T = \dfrac{2\pi\mathfrak{R}}{c} = 39$ million years.

Critical density, $\rho^* = \dfrac{3c^2}{4\pi\mathfrak{R}^2} = 6{\cdot}20 \,.\, 10^{-30}$ astronomical mass units per cm.^{3} or $9{\cdot}17 \,.\, 10^{-23}$ gr/cm.^{3}.

Critical radius of the Sun, $r^* = (L\mathfrak{R}^2)^{\frac{1}{3}} = 0{\cdot}558$ parsec.

Critical radius of our galaxy, $r^* = 840$ parsecs or 2,740 light-years. The instability of the Milky Way is thus seen to be even much more pronounced than was stated before.

For the globular cluster N.G.C. 6205, with a mass such as given on p. 174, we would have $r^* = 42$ l.y. only. But, as I have since learned, the mass of this cluster is probably some 20 or 30 times greater (generally the globular clusters consist not of tens but of hundreds of thousands of stars), so that $r^* = 120$ light-years. And since the semi-diameter is 80 l.y., the criterion of permanency, for this and other globular clusters, is still satisfied.

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